

RUMUS-RUMUS TRIGONOMETRI

B. Rumus Trigonometri Sudut Ganda dan Sudut Tengahan

A. Rumus Sudut Ganda

Yang dimaksud dengan sudut ganda adalah sudut 2α . Untuk mendapatkan rumus trigonometri untuk $\sin 2\alpha$, $\cos 2\alpha$ dan $\tan 2\alpha$, diperoleh dari rumus-rumus sebelumnya, yakni:

(1) Sudut $\sin 2\alpha$

$$\sin (\alpha + \beta) = \sin \alpha . \cos \beta + \cos \alpha . \sin \beta$$

$$\sin (\alpha + \alpha) = \sin \alpha . \cos \alpha + \cos \alpha . \sin \alpha$$

$$\sin 2\alpha = 2 . \sin \alpha . \cos \alpha \dots\dots\dots (1)$$

(2) Sudut $\cos 2\alpha$

$$\cos (\alpha + \beta) = \cos \alpha . \cos \beta - \sin \alpha . \sin \beta$$

$$\cos (\alpha + \alpha) = \cos \alpha . \cos \alpha - \sin \alpha . \sin \alpha$$

$$\cos 2\alpha = \cos^2 \alpha - \sin^2 \alpha \dots\dots\dots (2)$$

Rumus $\cos 2\alpha$ yang lain :

$$\cos 2\alpha = \cos^2 \alpha - \sin^2 \alpha$$

$$\cos 2\alpha = (1 - \sin^2 \alpha) - \sin^2 \alpha$$

$$\cos 2\alpha = 1 - 2\sin^2 \alpha \dots\dots\dots (3)$$

atau

$$\cos 2\alpha = \cos^2 \alpha - \sin^2 \alpha$$

$$\cos 2\alpha = \cos^2 \alpha - (1 - \cos^2 \alpha)$$

$$\cos 2\alpha = 2 . \cos^2 \alpha - 1 \dots\dots\dots (4)$$

(3) Sudut $\tan 2\alpha$

$$\tan (\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha . \tan \beta}$$

$$\tan (\alpha + \alpha) = \frac{\tan \alpha + \tan \alpha}{1 - \tan \alpha . \tan \alpha}$$

$$\tan 2\alpha = \frac{2 . \tan \alpha}{1 - \tan^2 \alpha} \dots\dots\dots (5)$$

Untuk lebih jelasnya, pelajirlah contoh soal berikut ini:

01. Tentukanlah nilai dari :

(a) $4 . \cos^2 67,5^0 - 4 \sin^2 67,5^0 + 6\sqrt{2}$

(b) $12\sqrt{3} \cos^2 15^0 - 6\sqrt{3}$

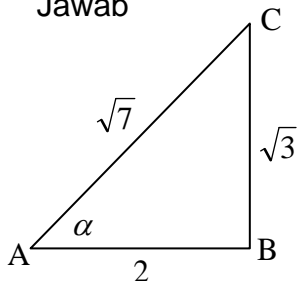
Jawab

$$\begin{aligned}
\text{(a)} \quad & 4 \cos^2 67,5^0 - 4 \sin^2 67,5^0 + 6 \sqrt{2} \\
& = 4(\cos^2 67,5^0 - \sin^2 67,5^0) + 6 \sqrt{2} \\
& = 4 \cdot \cos 2(67,5^0) + 6 \sqrt{2} \\
& = 4 \cdot \cos 135^0 + 6 \sqrt{2} \\
& = 4\left(-\frac{1}{2}\sqrt{2}\right) + 6 \sqrt{2} \\
& = -2\sqrt{2} + 6\sqrt{2} \\
& = 4\sqrt{2}
\end{aligned}$$

$$\begin{aligned}
\text{(b)} \quad & 12\sqrt{3} \cos^2 15^0 - 6\sqrt{3} \\
& = 6\sqrt{3} (2 \cos^2 15^0 - 1) \\
& = 6\sqrt{3} \cdot \cos 2(15^0) \\
& = 6\sqrt{3} \cdot \cos 30^0 \\
& = 6\sqrt{3} \cdot \left(\frac{1}{2}\sqrt{3}\right) \\
& = 9
\end{aligned}$$

02. Jika $\tan \alpha = \frac{1}{2}\sqrt{3}$ dan α sudut lancip, maka tentukanlah nilai $\sin 2\alpha$

Jawab



$$\begin{aligned}
AC^2 &= AB^2 + BC^2 \\
AC^2 &= 2^2 + (\sqrt{3})^2 \\
AC^2 &= 7 \\
\text{Jadi } AC &= \sqrt{7}
\end{aligned}$$

Sehingga : $\tan \alpha = \frac{1}{2}\sqrt{3}$

$$\sin \alpha = \frac{\sqrt{3}}{\sqrt{7}} = \frac{\sqrt{3}}{\sqrt{7}} \times \frac{\sqrt{7}}{\sqrt{7}} = \frac{\sqrt{21}}{7}$$

$$\cos \alpha = \frac{2}{\sqrt{7}} = \frac{2}{\sqrt{7}} \times \frac{\sqrt{7}}{\sqrt{7}} = \frac{2\sqrt{7}}{7}$$

Jadi $\sin 2\alpha = 2 \cdot \sin \alpha \cdot \cos \alpha$

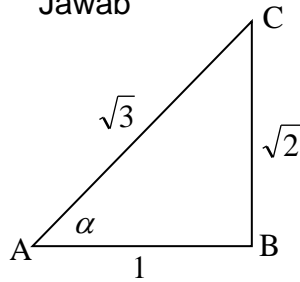
$$= 2\left(\frac{\sqrt{21}}{7}\right)\left(\frac{2\sqrt{7}}{7}\right)$$

$$= \frac{2 \cdot x \cdot \sqrt{21} \cdot x \cdot 2 \cdot x \cdot \sqrt{7}}{7 \cdot x \cdot 7}$$

$$= \frac{4}{7}\sqrt{3}$$

03. Jika $\cos \alpha = -\frac{1}{\sqrt{3}}$ dan $90^\circ < \alpha < 180^\circ$, maka tentukanlah nilai $\tan 2\alpha$

Jawab



$$\begin{aligned} BC^2 &= AC^2 - AB^2 \\ BC^2 &= (\sqrt{3})^2 - (1)^2 \\ BC^2 &= 2 \\ \text{Jadi } BC &= \sqrt{2} \end{aligned}$$

$$\begin{aligned} \text{Sehingga : } \cos \alpha &= -\frac{1}{\sqrt{3}} \\ \tan \alpha &= -\frac{\sqrt{2}}{1} = -\sqrt{2} \end{aligned}$$

$$\begin{aligned} \text{Jadi } \tan 2\alpha &= \frac{2 \cdot \tan \alpha}{1 - \tan^2 \alpha} \\ &= \frac{2(-\sqrt{2})}{1 - (-\sqrt{2})^2} \\ &= \frac{-2\sqrt{2}}{1 - 2} \\ &= 2\sqrt{2} \end{aligned}$$

04. Buktikanlah bahwa $\frac{1 - \tan^2 \alpha}{1 + \tan^2 \alpha} = \cos 2\alpha$

Jawab

$$\begin{aligned} \text{Ruas Kiri} &= \frac{1 - \tan^2 \alpha}{1 + \tan^2 \alpha} \\ &= \frac{1 - \frac{\sin^2 \alpha}{\cos^2 \alpha}}{1 + \frac{\sin^2 \alpha}{\cos^2 \alpha}} \\ &= \frac{\frac{\cos^2 \alpha}{\cos^2 \alpha} - \frac{\sin^2 \alpha}{\cos^2 \alpha}}{\frac{\cos^2 \alpha}{\cos^2 \alpha} + \frac{\sin^2 \alpha}{\cos^2 \alpha}} \\ &= \frac{\cos^2 \alpha - \sin^2 \alpha}{\cos^2 \alpha + \sin^2 \alpha} \\ &= \frac{\cos 2\alpha}{1} \end{aligned}$$

$$= \cos 2\alpha$$

$$= \text{ruas kanan}$$

05. Jika α sudut lancip yang memenuhi $2.\cos^2 \alpha = 1 + 2.\sin 2\alpha$, maka tentukanlah nilai $\tan 4\alpha$

Jawab

$$2.\cos^2 \alpha = 1 + 2.\sin 2\alpha$$

$$2.\cos^2 \alpha - 1 = 2.\sin 2\alpha$$

$$\cos 2\alpha = 2.\sin 2\alpha$$

$$\frac{\sin 2\alpha}{\cos 2\alpha} = \frac{1}{2}$$

$$\tan 2\alpha = 1/2$$

$$\text{Sehingga } \tan 4\alpha = \tan 2(2\alpha)$$

$$= \frac{2.\tan 2\alpha}{1 - \tan^2 2\alpha}$$

$$= \frac{2.(1/2)}{1 - (1/2)^2}$$

$$= \frac{1}{1 - \frac{1}{4}}$$

$$= \frac{1}{3/4}$$

$$= \frac{4}{3}$$

Yang dimaksud dengan sudut tengahan adalah sudut $\frac{1}{2}\alpha$. Untuk mendapatkan rumus

trigonometri untuk $\sin \frac{1}{2}\alpha$, $\cos \frac{1}{2}\alpha$ dan $\tan \frac{1}{2}\alpha$, diperoleh dari rumus-rumus sebelumnya, yakni:

$$\text{Karena } \cos 2\alpha = 1 - 2\sin^2 \alpha \text{ maka } \cos \alpha = 1 - 2\sin^2 \frac{1}{2}\alpha$$

$$2\sin^2 \frac{1}{2}\alpha = 1 - \cos \alpha$$

$$\sin \frac{1}{2}\alpha = \pm \sqrt{\frac{1 - \cos \alpha}{2}} \dots\dots\dots (6)$$

Karena $\cos 2\alpha = 2\cos^2\alpha - 1$ maka $\cos \alpha = 2\cos^2 \frac{1}{2}\alpha - 1$

$$2\cos^2 \frac{1}{2}\alpha = 1 + \cos \alpha$$

$$\cos \frac{1}{2}\alpha = \pm \sqrt{\frac{1 + \cos \alpha}{2}} \dots\dots\dots (7)$$

Karena $\tan \alpha = \frac{\sin \alpha}{\cos \alpha}$ maka $\tan \frac{1}{2}\alpha = \frac{\sin \frac{1}{2}\alpha}{\cos \frac{1}{2}\alpha}$

$$\tan \frac{1}{2}\alpha = \pm \frac{\sqrt{\frac{1 - \cos \alpha}{2}}}{\sqrt{\frac{1 + \cos \alpha}{2}}}$$

$$\tan \frac{1}{2}\alpha = \pm \sqrt{\frac{1 - \cos \alpha}{1 + \cos \alpha}} \dots\dots\dots (8)$$

Dari rumus (8) dapat dikembangkan rumus :

$$\tan \frac{1}{2}\alpha = \pm \sqrt{\frac{1 - \cos \alpha}{1 + \cos \alpha}}$$

$$\tan \frac{1}{2}\alpha = \pm \sqrt{\left(\frac{1 - \cos \alpha}{1 + \cos \alpha}\right)\left(\frac{1 - \cos \alpha}{1 - \cos \alpha}\right)}$$

$$\tan \frac{1}{2}\alpha = \pm \sqrt{\frac{(1 - \cos \alpha)^2}{1 - \cos^2 \alpha}}$$

$$\tan \frac{1}{2}\alpha = \pm \sqrt{\frac{(1 - \cos \alpha)^2}{\sin^2 \alpha}}$$

$$\tan \frac{1}{2}\alpha = \frac{1 - \cos \alpha}{\sin \alpha} \dots\dots\dots (9)$$

Atau

$$\tan \frac{1}{2}\alpha = \pm \sqrt{\frac{1 - \cos \alpha}{1 + \cos \alpha}}$$

$$\tan \frac{1}{2}\alpha = \pm \sqrt{\left(\frac{1 - \cos \alpha}{1 + \cos \alpha}\right)\left(\frac{1 + \cos \alpha}{1 + \cos \alpha}\right)}$$

$$\tan \frac{1}{2}\alpha = \pm \sqrt{\frac{1 - \cos^2 \alpha}{(1 + \cos \alpha)^2}}$$

$$\tan \frac{1}{2}\alpha = \pm \sqrt{\frac{\sin^2 \alpha}{(1 + \cos \alpha)^2}}$$

$$\tan \frac{1}{2}\alpha = \frac{\sin \alpha}{1 + \cos \alpha} \dots\dots\dots (10)$$

Untuk lebih jelasnya, pelajailah contoh soal berikut ini:

06. Tentukanlah nilai dari :

(a) $\cos 112,5^\circ$

(b) $\tan 22,5^\circ$

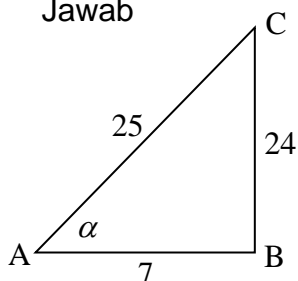
Jawab

$$\begin{aligned} \text{(a) } \cos 112,5^\circ &= \cos \frac{1}{2}(225^\circ) \\ &= -\sqrt{\frac{1 + \cos 225^\circ}{2}} \\ &= -\sqrt{\frac{1 - \frac{1}{2}\sqrt{2}}{2}} \\ &= -\sqrt{\frac{2 - \sqrt{2}}{4}} \\ &= -\frac{1}{2}\sqrt{2 - \sqrt{2}} \end{aligned}$$

$$\begin{aligned} \text{(b) } \tan 22,5^\circ &= \tan \frac{1}{2}(45^\circ) \\ &= \frac{1 - \cos 45^\circ}{\sin 45^\circ} \\ &= \frac{1 - \frac{1}{2}\sqrt{2}}{\frac{1}{2}\sqrt{2}} \\ &= \frac{2 - \sqrt{2}}{\sqrt{2}} \\ &= \sqrt{2} - 1 \end{aligned}$$

07. Jika $\cos \alpha = 7/25$ dan $270^\circ < \alpha < 360^\circ$ maka tentukanlah nilai $\tan \frac{1}{2} \alpha = \dots$

Jawab



$$\begin{aligned} BC^2 &= AC^2 - AB^2 \\ BC^2 &= (25)^2 - (7)^2 \\ AC^2 &= 576 \\ \text{Jadi } AC &= 24 \end{aligned}$$

Sehingga : $\cos \alpha = \frac{7}{25}$ dan $\sin \alpha = -\frac{24}{25}$

$$\text{Jadi : } \tan \frac{1}{2} \alpha = -\frac{1 - \cos \alpha}{\sin \alpha}$$

$$\tan \frac{1}{2} \alpha = -\frac{1 - \frac{7}{25}}{-\frac{24}{25}}$$

$$\tan \frac{1}{2} \alpha = -\frac{25 - 7}{-24}$$

$$\tan \frac{1}{2} \alpha = -\frac{18}{24}$$

$$\tan \frac{1}{2} \alpha = -\frac{3}{4}$$

Dari uraian di atas dapat pula diturunkan Rumus trigonometri untuk Sudut Yang Lain, yakni :

$$\begin{aligned} \sin 3\alpha &= \sin (2\alpha + \alpha) \\ &= \sin 2\alpha \cdot \cos \alpha + \cos 2\alpha \cdot \sin \alpha \\ &= (2\sin \alpha \cdot \cos \alpha) \cdot \cos \alpha + (1 - 2\sin^2 \alpha) \cdot \sin \alpha \\ &= 2\sin \alpha \cdot \cos^2 \alpha + \sin \alpha - 2\sin^3 \alpha \\ &= 2\sin \alpha \cdot (1 - \sin^2 \alpha) + \sin \alpha - 2\sin^3 \alpha \\ &= 2\sin \alpha - 2\sin^3 \alpha + \sin \alpha - 2\sin^3 \alpha \\ &= -4\sin^3 \alpha + 3\sin \alpha \end{aligned}$$

$$\begin{aligned} \cos 3\alpha &= \cos (2\alpha + \alpha) \\ &= \cos 2\alpha \cdot \cos \alpha - \sin 2\alpha \cdot \sin \alpha \\ &= (2\cos^2 \alpha - 1) \cos \alpha - 2\sin \alpha \cdot \cos \alpha \cdot \sin \alpha \\ &= 2\cos^3 \alpha - \cos \alpha - 2\sin^2 \alpha \cdot \cos \alpha \\ &= 2\cos^3 \alpha - \cos \alpha - 2(1 - \cos^2 \alpha) \cos \alpha \\ &= 2\cos^3 \alpha - \cos \alpha - 2\cos \alpha + 2\cos^3 \alpha \\ &= 4\cos^3 \alpha - 3\cos \alpha \end{aligned}$$

Selain dua rumus di atas, dengan cara yang sama dapat juga diturunkan rumus-rumus yang lain