M A T R I K S

G. Menyelesaikan Sistem Persamaan Linier Dengan Matriks

(1) Sistem Persamaan Linier dua Variabel

Salah satu diantara penggunaan invers matriks adalah untuk menyelesaikan sistim persamaan linier. Tentu saja teknik penyelesaiannya dengan aturan persamaan matriks, yaitu :

Jika $\begin{cases} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{cases}$ maka $\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{ad - bc}\begin{bmatrix} b_2 & -b_1 \\ -a_2 & a_1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Selain dengan persamaan matriks, teknik menyelesaikan sistem persamaan linier juga dapat dilakukan dengan determinan matriks. Aturan dengan cara ini adalah :

Jika matriks A = $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ maka det(A) = $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ = ad – bc. sehingga

Jika $\begin{cases} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{cases}$ maka D = $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - b_1a_2$

$$D_x = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} = c_1b_2 - b_1c_2$$

$$D_y = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} = a_1c_2 - c_1a_2$$

Maka x = $\dfrac{D_x}{D}$ dan y = $\dfrac{D_y}{D}$

Untuk lebih jelanya, ikutolah contoh soal berikut ini:

01. Tentukan penyelesaian sistem persamaan 2x – 3y = 8 dan x + 2y = –3 dengan metoda:
    (a) Invers matriks (b) Determinan

    Jawab

    2x – 3y = 8
    x + 2y = –3
    maka $\begin{bmatrix} 2 & -3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ -3 \end{bmatrix}$

(a) Dengan metoda invers matriks diperoleh

$$\begin{bmatrix} 2 & -3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ -3 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{4-(-3)}\begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 8 \\ -3 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 8 \\ -3 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 16-9 \\ -8-6 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 7 \\ -14 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

Jadi x = 1 dan y = –2

(b) Dengan metoda determinan matriks diperoleh

$$D = \begin{vmatrix} 2 & -3 \\ 1 & 2 \end{vmatrix} = (2)(2) - (-3)(1) = 4 + 3 = 7$$

$$D_x = \begin{vmatrix} 8 & -3 \\ -3 & 2 \end{vmatrix} = (8)(2) - (-3)(-3) = 16 - 9 = 7$$

$$D_y = \begin{vmatrix} 2 & 8 \\ 1 & -3 \end{vmatrix} = (2)(-3) - (8)(1) = -6 - 8 = -14$$

Maka $x = \dfrac{D_x}{D} = \dfrac{7}{7} = 1$

$y = \dfrac{D_y}{D} = \dfrac{-14}{7} = 2$

02. Tentukan penyelesaian sistem persamaan $y = \frac{1}{2}x + 5$ dan $x + 6 = \frac{2}{3}y$ dengan metoda:

(a) Invers matriks (b) Determinan

Jawab

$y = \frac{1}{2}x + 5 \;\Big|\; (2) \longrightarrow 2y = x + 10 \longrightarrow -x + 2y = 10$

$x + 6 = \frac{2}{3}y \;\Big|\; (3) \longrightarrow 3x + 18 = 2y \longrightarrow 3x - 2y = -18$

–x + 2y = 10

3x – 2y = –18

Maka

$$D = \begin{vmatrix} -1 & 2 \\ 3 & -2 \end{vmatrix} = (–1)(–2) – (2)(3) = 2 – 6 = –4$$

$$D_x = \begin{vmatrix} 10 & 2 \\ -18 & -2 \end{vmatrix} = (10)(–2) – (2)(–18) = –20 – (–36) = 16$$

$$D_y = \begin{vmatrix} -1 & 10 \\ 3 & -18 \end{vmatrix} = (–1)(–18) – (10)(3) = 18 – 30 = –12$$

Maka $x = \dfrac{D_x}{D} = \dfrac{16}{-4} = -4$

$y = \dfrac{D_y}{D} = \dfrac{-12}{-4} = 3$

03. Tentukan penyelesaian sistem persamaan 2y – 3x = –4 dan 2x + y = 5 dengan metoda:

(a) Invers matriks

(b) Determinan

Jawab

2y – 3x = –4 $\longrightarrow$ 3x – 2y = 4

x + 2y = –3 $\longrightarrow$ 2x + y = 5 maka $\begin{bmatrix} 3 & -2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$

(a) Dengan metoda invers matriks diperoleh

$$\begin{bmatrix} 3 & -2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{3-(-4)}\begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 4+10 \\ -8+15 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 14 \\ 7 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Jadi x = 2 dan y = 1

(b) Dengan metoda determinan matriks diperoleh

$$D = \begin{vmatrix} 3 & -2 \\ 2 & 1 \end{vmatrix} = (3)(1) - (-2)(2) = 3 + 4 = 7$$

$$D_x = \begin{vmatrix} 4 & -2 \\ 5 & 1 \end{vmatrix} = (4)(1) - (-2)(5) = 4 + 10 = 14$$

$$D_y = \begin{vmatrix} 3 & 4 \\ 2 & 5 \end{vmatrix} = (3)(5) - (4)(2) = 15 - 8 = 7$$

Maka $x = \dfrac{D_x}{D} = \dfrac{14}{7} = 2$

$y = \dfrac{D_y}{D} = \dfrac{7}{7} = 1$

(2) Sistem Persamaan Linier Tiga Variabel.

Sepeti halnya pada sistem persamaan linier dua variabel, menyelesaikan sistem persamaan linier tiga variabel dengan matriks juga terdiri dari dua cara, yakni dengan menggunakan determinan matriks dan dengan menggunakan aturan invers perkalian matriks. Berikut ini akan diuraikan masing masing cara tersebut.

Aturan menyelesaikan sistem persamaan linier menggunakan determinan matriks adalah dengan menentukan terlebih dahulu matriks koefisien dari sistem persamaan itu.

Selanjutnya ditentukan empat nilai determinan sebagai berikut:

(1) D yakni determinan matriks koefisien

(2) D_x yakni determinan matriks koefisien dengan koefisien x diganti konstanta

(3) D_y yakni determinan matriks koefisien dengan koefisien y diganti konstanta

(4) D_z yakni determinan matriks koefisien dengan koefisien z diganti konstanta

Rumus masing-masingnya adalah sebagai berikut :

Jika $\begin{cases} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \\ a_3x + b_3y + c_3z = d_3 \end{cases}$ diperoleh nilai determinan :

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1.b_2.c_3 + b_1.c_2.a_3 + c_1.a_2.b_3 - c_1.b_2.a_3 - a_1.c_2.b_3 - b_1.a_2.c_3$$

$$D_x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix} = d_1.b_2.c_3 + b_1.c_2.d_3 + c_1.d_2.b_3 - c_1.b_2.d_3 - d_1.c_2.b_3 - b_1.d_2.c_3$$

$$D_y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} = a_1.d_2.c_3 + d_1.c_2.a_3 + c_1.a_2.d_3 - c_1.d_2.a_3 - a_1.c_2.d_3 - d_1.a_2.c_3$$

$$D_z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix} = a_1.b_2.d_3 + b_1.d_2.a_3 + d_1.a_2.b_3 - d_1.b_2.a_3 - a_1.d_2.b_3 - b_1.a_2.d_3$$

Sehingga nilai $x = \frac{D_x}{D}$, $y = \frac{D_y}{D}$ dan $z = \frac{D_z}{D}$

Untuk lebih jelasnya, ikutilah contoh soal berikut ini:

01. Tentukanlah himpunan penyelesaian sistem persamaan linier

$$\left.\begin{aligned} 2x - 3y + 2z &= -3 \\ x + 2y + z &= 2 \\ 2x - y + 3z &= 1 \end{aligned}\right\} \text{ dengan menggunakan metoda determinan}$$

Jawab

$$D = \begin{vmatrix} 2 & -3 & 2 \\ 1 & 2 & 1 \\ 2 & -1 & 3 \end{vmatrix} = \left|\begin{array}{ccc:cc} 2 & -3 & 2 & 2 & -3 \\ 1 & 2 & 1 & 1 & 2 \\ 2 & -1 & 3 & 2 & -1 \end{array}\right|$$

$D = (2)(2)(3) + (-3)(1)(2) + (2)(1)(-1) - (2)(2)(2) - (2)(1)(-1) - (-3)(1)(3)$

$D = 12 - 6 - 2 - 8 + 2 + 9$

$D = 7$

$$D_x = \begin{vmatrix} -3 & -3 & 2 \\ 2 & 2 & 1 \\ 1 & -1 & 3 \end{vmatrix} = \left|\begin{array}{cc:ccc} -3 & -3 & 2 & -3 & -3 \\ 2 & 2 & 1 & 2 & 2 \\ 1 & -1 & 3 & 1 & -1 \end{array}\right|$$

$D_x = (-3)(2)(3) + (-3)(1)(1) + (2)(2)(-1) - (2)(2)(1) - (-3)(1)(-1) - (-3)(2)(3)$

$D_x = -18 - 3 - 4 - 4 - 3 + 18$

$D_x = -14$

$$D_y = \begin{vmatrix} 2 & -3 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 3 \end{vmatrix} = \left|\begin{array}{cc:ccc} 2 & -3 & 2 & 2 & -3 \\ 1 & 2 & 1 & 1 & 2 \\ 2 & 1 & 3 & 2 & 1 \end{array}\right|$$

$D_y = (2)(2)(3) + (-3)(1)(2) + (2)(1)(1) - (2)(2)(2) - (2)(1)(1) - (-3)(1)(3)$

$D_y = 12 - 6 + 2 - 8 - 2 + 9$

$D_y = 7$

$$D_z = \begin{vmatrix} 2 & -3 & -3 \\ 1 & 2 & 2 \\ 2 & -1 & 1 \end{vmatrix} = \left|\begin{array}{cc:ccc} 2 & -3 & -3 & 2 & -3 \\ 1 & 2 & 2 & 1 & 2 \\ 2 & -1 & 1 & 2 & -1 \end{array}\right|$$

$D_z = (2)(2)(1) + (–3)(2)(2) + (–3)(1)(–1) – (–3)(2)(2) – (2)(2)(–1) – (–3)(1)(1)$
$D_z = 4 – 12 + 3 + 12 + 4 + 3$
$D_z = 14$

Jadi $x = \dfrac{D_x}{D} = \dfrac{-14}{7} = -2$

$y = \dfrac{D_y}{D} = \dfrac{7}{7} = 1$

$z = \dfrac{D_z}{D} = \dfrac{14}{7} = 2$

02. Tentukanlah himpunan penyelesaian sistem persamaan linier

$$\left.\begin{aligned} x + 2y + z &= 2 \\ x - y - 2z &= -1 \\ x + y - z &= 3 \end{aligned}\right\}$$ dengan menggunakan metoda determinan

Jawab

$$D = \begin{vmatrix} 1 & 2 & 1 \\ 1 & -1 & -2 \\ 1 & 1 & -1 \end{vmatrix} = \left|\begin{array}{cc:ccc} 1 & 2 & 1 & 1 & 2 \\ 1 & -1 & -2 & 1 & -1 \\ 1 & 1 & -1 & 1 & 1 \end{array}\right|$$

$D = (1)(–1)(–1) + (2)(–2)(1) + (1)(1)(1) – (1)(–1)(1) – (1)(–2)(1) – (2)(1)(–1)$
$D = 1 – 4 + 1 + 1 + 2 + 2$
$D = 3$

$$D_x = \begin{vmatrix} 2 & 2 & 1 \\ -1 & -1 & -2 \\ 3 & 1 & -1 \end{vmatrix} = \left|\begin{array}{cc:ccc} 2 & 2 & 1 & 2 & 2 \\ -1 & -1 & -2 & -1 & -1 \\ 3 & 1 & -1 & 3 & 1 \end{array}\right|$$

$D_x = (2)(–1)(–1) + (2)(–2)(3) + (1)(–1)(1) – (1)(–1)(3) – (2)(–2)(1) – (2)(–1)(–1)$
$D_x = 2 – 12 – 1 + 3 + 4 – 2$
$D_x = –6$

$$D_y = \begin{vmatrix} 1 & 2 & 1 \\ 1 & -1 & -2 \\ 1 & 3 & -1 \end{vmatrix} = \left|\begin{array}{cc:ccc} 1 & 2 & 1 & 1 & 2 \\ 1 & -1 & -2 & 1 & -1 \\ 1 & 3 & -1 & 1 & 3 \end{array}\right|$$

$D_y = (1)(–1)(–1) + (2)(–2)(1) + (1)(1)(3) – (1)(–1)(1) – (1)(–2)(3) – (2)(1)(–1)$
$D_y = 1 – 4 + 3 + 1 + 6 + 2$
$D_y = 9$

$$D_z = \begin{vmatrix} 1 & 2 & 2 \\ 1 & -1 & -1 \\ 1 & 1 & 3 \end{vmatrix} = \left|\begin{array}{cc:ccc} 1 & 2 & 2 & 1 & 2 \\ 1 & -1 & -1 & 1 & -1 \\ 1 & 1 & 3 & 1 & 1 \end{array}\right|$$

D_z = (1)(–1)(3) + (2)(–1)(1) + (2)(1)(1) – (2)(–1)(1) – (1)(–1)(1) – (2)(1)(3)
D_z = –3 – 2 + 2 + 2 + 1 – 6
D_z = –6

Jadi $x = \dfrac{D_x}{D} = \dfrac{-6}{3} = -2$

$y = \dfrac{D_y}{D} = \dfrac{9}{3} = 3$

$z = \dfrac{D_z}{D} = \dfrac{-6}{3} = -2$

03. Tentukanlah himpunan penyelesaian sistem persamaan linier

$\left.\begin{array}{l} x - 2y = -3 \\ y + z = 1 \\ 2x + z = 1 \end{array}\right\}$ dengan menggunakan metoda determinan

Jawab

$$D = \begin{vmatrix} 1 & -2 & 0 \\ 0 & 1 & 1 \\ 2 & 0 & 1 \end{vmatrix} = \left|\begin{array}{ccc:cc} 1 & -2 & 0 & 1 & -2 \\ 0 & 1 & 1 & 0 & 1 \\ 2 & 0 & 1 & 2 & 0 \end{array}\right|$$

D = (1)(1)(1) + (–2)(1)(2) + (0)(0)(0) – (0)(1)(2) – (1)(1)(0) – (–2)(0)(1)
D = (1) + (–4) + (0) – (0) – (0) – (0)
D = 1 – 4 + 0 + 0 + 0 + 0
D = –3

$$D_x = \begin{vmatrix} -3 & -2 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix} = \left|\begin{array}{cc:ccc} -3 & -2 & 0 & -3 & -2 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 \end{array}\right|$$

D_x = (–3)(1)(1) + (–2)(1)(1) + (0)(1)(0) – (0)(1)(1) – (–3)(1)(0) – (–2)(1)(1)
D_x = (–3) + (–2) + (0) – (0) – (0) – (–2)
D_x = –3 – 2 + 0 – 0 – 0 + 2
D_x = –3

$$D_y = \begin{vmatrix} 1 & -3 & 0 \\ 0 & 1 & 1 \\ 2 & 1 & 1 \end{vmatrix} = \left|\begin{array}{cc:ccc} 1 & -3 & 0 & 1 & -3 \\ 0 & 1 & 1 & 0 & 1 \\ 2 & 1 & 1 & 2 & 1 \end{array}\right|$$

D_y = (1)(1)(1) + (–3)(1)(2) + (0)(0)(1) – (0)(1)(2) – (1)(1)(1) – (–3)(0)(1)
D_y = (1) + (–6) + (0) – (0) – (1) – (0)
D_y = 1 – 6 + 0 + 0 – 1 – 0
D_y = –6

$$D_z = \begin{vmatrix} 1 & -2 & -3 \\ 0 & 1 & 1 \\ 2 & 0 & 1 \end{vmatrix} = \left|\begin{array}{ccc:cc} 1 & -2 & -3 & 1 & -2 \\ 0 & 1 & 1 & 0 & 1 \\ 2 & 0 & 1 & 2 & 0 \end{array}\right|$$

$D_z = (1)(1)(1) + (-2)(1)(2) + (-3)(0)(0) - (-3)(1)(2) - (1)(1)(0) - (-2)(0)(1)$

$D_z = (1) + (-4) + (0) - (-6) - (0) - (0)$

$D_z = 1 - 4 + 0 + 6 - 0 - 0$

$D_z = 3$

Jadi $x = \dfrac{D_x}{D} = \dfrac{-3}{-3} = 1$

$y = \dfrac{D_y}{D} = \dfrac{-6}{-3} = 2$

$z = \dfrac{D_z}{D} = \dfrac{3}{-3} = -1$

Untuk menyelesaikan sistem persamaan linier tiga variabel dengan invers matriks dilakukan dengan tahapan berikut ini :

Jika $\begin{cases} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \\ a_3x + b_3y + c_3z = d_3 \end{cases}$ diperoleh persamaan matriks

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} . \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

$$A \quad . \quad B = C$$

$$B = A^{-1} . C$$

Karena A adalah matriks koefisien berordo (3 x 3) maka A^{-1} adalah invers perkalian matriks berordo (3 x 3). Berikut ini tatacara menentukan invers matriks ordo (3 x 3)

Misalkan $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$ maka langkah-langkah menentukan invers matriks adalah sebagai berikut :

1. Menentukan minor matriks A untuk baris p dan kolom q (M_{pq})

$M_{11} = \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} = b_2c_3 - c_2b_3$ $\qquad$ $M_{12} = \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} = a_2c_3 - c_2a_3$

$M_{13} = \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} = a_2b_3 - b_2a_3$ $\qquad$ $M_{21} = \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} = b_1c_3 - c_1b_3$

$M_{22} = \begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} = a_1c_3 - c_1a_3$ $\qquad$ $M_{23} = \begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix} = a_1b_3 - b_1a_3$

$M_{31} = \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} = b_1c_2 - c_1b_2$ $M_{32} = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} = a_1c_2 - c_1a_2$

$M_{33} = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - b_1a_2$

2. Menentukan kofaktor matriks A

Kofaktor matriks A baris ke-p kolam ke-q dilambangkan C_{pq} ditentukan dengan rumus :

$$C_{pq} = (-1)^{p+q} M_{pq}$$

Sehingga diperoleh matriks kofaktor C sebagai berikut :

$$C = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{22} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

3. Menentukan determinan matriks A

Determinan matriks A ditulis ditentukan dengan menggunakan kofaktor C_{pq}

dengan rumus : $\det(A) = a_1C_{11} - b_1C_{12} + c_1C_{13}$

$\det(A) = a_2C_{21} - b_2C_{22} + c_2C_{23}$

$\det(A) = a_3C_{31} - b_3C_{32} + c_3C_{33}$

4. Menentukan matriks adjoint A, yakni transpose dari kofaktor matriks A, atau dirumuskan :

$$\text{Adj } A = C^t$$

5. Menentukan invers matriks A dengan rumus :

$$A^{-1} = \frac{1}{\det(A)} \text{ adj } A$$

Untuk lebih jelasnya, ikutilah contoh soal berikut ini:

04. Tentukanlah himpunan penyelesaian sistem persamaan linier

$$\left.\begin{aligned} x + y - 2z &= 2 \\ x - 3y + z &= -3 \\ x - y + z &= 1 \end{aligned}\right\} \text{ dengan menggunakan metoda inver matriks}$$

Jawab

Matriks koefisien untuk sistem persamaan linier diataas adalah :

$$A = \begin{bmatrix} 1 & 1 & -2 \\ 1 & -3 & 1 \\ 1 & -1 & 1 \end{bmatrix} \text{ sehingga}$$

Pertama akan ditentukan minor matriks, yaitu

$M_{11} = \begin{vmatrix} -3 & 1 \\ -1 & 1 \end{vmatrix}$ = (–3)(1) – (1)(–1) = –3 + 1 = –2

$M_{12} = \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}$ = (1)(1) – (1)(1) = 1 – 1 = 0

$M_{13} = \begin{vmatrix} 1 & -3 \\ 1 & -1 \end{vmatrix}$ = (1)(–1) – (–3)(1) = –1 + 3 = 2

$M_{21} = \begin{vmatrix} 1 & -2 \\ -1 & 1 \end{vmatrix}$ = (1)(1) – (–2)(–1) = 1 – 2 = –1

$M_{22} = \begin{vmatrix} 1 & -2 \\ 1 & 1 \end{vmatrix}$ = (1)(1) – (–2)(1) = 1 + 2 = 3

$M_{23} = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}$ = (1)(–1) – (1)(1) = –1 – 1 = –2

$M_{31} = \begin{vmatrix} 1 & -2 \\ -3 & 1 \end{vmatrix}$ = (1)(1) – (–2)(–3) = 1 – 6 = –5

$M_{32} = \begin{vmatrix} 1 & -2 \\ 1 & 1 \end{vmatrix}$ = (1)(1) – (–2)(1) = 1 + 2 = 3

$M_{33} = \begin{vmatrix} 1 & 1 \\ 1 & -3 \end{vmatrix}$ = (1)(–3) – (1)(1) = –3 – 1 = –4

Kemudian menentukan kofaktor matriks

$C_{11} = (-1)^{1+1}\ M_{11}$ = (1)(–2) = –2

$C_{12} = (-1)^{1+2}\ M_{12}$ = (–1)(0) = 0

$C_{13} = (-1)^{1+3}\ M_{13}$ = (1)(2) = 2

$C_{21} = (-1)^{2+1}\ M_{21}$ = (–1)(–1) = 1

$C_{22} = (-1)^{2+2}\ M_{22}$ = (1)(3) = 3

$C_{23} = (-1)^{2+3}\ M_{23}$ = (–1)(–2) = 2

$C_{31} = (-1)^{3+1}\ M_{31}$ = (1)(–5) = –5

$C_{32} = (-1)^{3+2}\ M_{32}$ = (–1)(3) = –3

$C_{33} = (-1)^{3+3}\ M_{33}$ = (1)(4) = 4

Matriks kofaktornya : $\quad C = \begin{bmatrix} -2 & 0 & 2 \\ 1 & 3 & 2 \\ -5 & -3 & -4 \end{bmatrix}$

Setelah itu menentukan Determinan matriks menggunakan ekspansi baris pertama

det(A) = $a_{11} C_{11} + a_{12} C_{12} + a_{13} C_{13}$ = (–2)(1) + (0)(1) + (2)(–2) = –6

Berikutnya menentukan Adjoint matriks dari matriks kofaktor

Jika kofaktor $C = \begin{bmatrix} -2 & 0 & 2 \\ 1 & 3 & 2 \\ -5 & -3 & -4 \end{bmatrix}$ maka adjoinnya $\text{adj(A)} = \begin{bmatrix} -2 & 1 & -5 \\ 0 & 3 & -3 \\ 2 & 2 & -4 \end{bmatrix}$

Akhirnya dimasukkan kedalam rumus invers matriks:

$A^{-1} = \dfrac{1}{\det(A)}$ adj(A) diperoleh $\quad A^{-1} = \dfrac{1}{-6} \begin{bmatrix} -2 & 1 & -5 \\ 0 & 3 & -3 \\ 2 & 2 & -4 \end{bmatrix}$

Setelah mendapatkan invers matriks, maka system persamaan linier diatas dielesaikan dengan persamaan matriks sebagai berikut :

$$\begin{bmatrix} 1 & 1 & -2 \\ 1 & -3 & 1 \\ 1 & -1 & 1 \end{bmatrix} . \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{-6} \begin{bmatrix} -2 & 1 & -5 \\ 0 & 3 & -3 \\ 2 & 2 & -4 \end{bmatrix} . \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{-6} \begin{bmatrix} -12 \\ -12 \\ -6 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$$

Jadi diperoleh nilai x = 2, y = 2 dan z = –1

05. Tentukanlah himpunan penyelesaian sistem persamaan linier

$$\left.\begin{aligned} 2x + 2y + 3z &= 4 \\ -x + y + 3z &= 1 \\ -2x + y + 2z &= -3 \end{aligned}\right\}$$ dengan menggunakan metoda inver matriks

Jawab

Matriks koefisien untuk sistem persamaan linier diataas adalah :

A = $\begin{bmatrix} 2 & 2 & 3 \\ -1 & 1 & 3 \\ -2 & 1 & 2 \end{bmatrix}$ sehingga

Pertama akan ditentukan minor matriks, yaitu

$M_{11} = \begin{vmatrix} 1 & 3 \\ 1 & 2 \end{vmatrix}$ = (1)(2) – (1)(3) = 2 – 3 = –1

$M_{12} = \begin{vmatrix} -1 & 3 \\ -2 & 2 \end{vmatrix}$ = (–1)(2) – (3)(–2) = –2 + 6 = 4

$M_{13} = \begin{vmatrix} -1 & 1 \\ -2 & 1 \end{vmatrix}$ = (–1)(1) – (1)(–2) = –1 + 2 = 1

$M_{21} = \begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix}$ = (2)(2) – (3)(1) = 4 – 3 = 1

$M_{22} = \begin{vmatrix} 2 & 3 \\ -2 & 2 \end{vmatrix}$ = (2)(2) – (3)(–2) = 4 + 6 = 10

$M_{23} = \begin{vmatrix} 2 & 2 \\ -2 & 1 \end{vmatrix}$ = (2)(1) – (2)(–2) = 2 + 4 = 6

$M_{31} = \begin{vmatrix} 2 & 3 \\ 1 & 3 \end{vmatrix}$ = (2)(3) – (3)(1) = 6 – 3 = 3

$M_{32} = \begin{vmatrix} 2 & 3 \\ -1 & 3 \end{vmatrix}$ = (2)(3) – (3)(–1) = 6 + 3 = 9

$M_{33} = \begin{vmatrix} 2 & 2 \\ -1 & 1 \end{vmatrix}$ = (2)(1) – (2)(–1) = 2 + 2 = 4

Kemudian menentukan kofaktor matriks

$C_{11} = (-1)^{1+1}\ M_{11}$ = (1)(–1) = –1

$C_{12} = (-1)^{1+2}\ M_{12}$ = (–1)(4) = –4

$C_{13} = (-1)^{1+3}\ M_{13}$ = (1)(1) = 1

$C_{21} = (-1)^{2+1}\ M_{21}$ = (–1)(1) = –1

$C_{22} = (-1)^{2+2}\ M_{22}$ = (1)(10) = 10

$C_{23} = (-1)^{2+3}\ M_{23}$ = (–1)(6) = –6

$C_{31} = (-1)^{3+1}\ M_{31} = (1)(3) = 3$

$C_{32} = (-1)^{3+2}\ M_{32} = (-1)(9) = -9$

$C_{33} = (-1)^{3+3}\ M_{33} = (1)(4) = 4$

Matriks kofaktornya : $C = \begin{bmatrix} -1 & -4 & 1 \\ -1 & 10 & -6 \\ 3 & -9 & 4 \end{bmatrix}$

Setelah itu menentukan Determinan matriks menggunakan ekspansi baris pertama

det(A) = $a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$ = (2)(–1) + (2)(–4) + (3)(1) = –7

Berikutnya menentukan Adjoint matriks dari matriks kofaktor

Jika kofaktor $C = \begin{bmatrix} -1 & -4 & 1 \\ -1 & 10 & -6 \\ 3 & -9 & 4 \end{bmatrix}$ maka adjoinnya $\text{adj}(A) = \begin{bmatrix} -1 & -1 & 3 \\ -4 & 10 & -9 \\ 1 & -6 & 4 \end{bmatrix}$

Akhirnya dimasukkan kedalam rumus invers matriks:

$A^{-1} = \dfrac{1}{\det(A)}\ \text{adj}(A)$ diperoleh $A^{-1} = \dfrac{1}{-7}\begin{bmatrix} -1 & -1 & 3 \\ -4 & 10 & -9 \\ 1 & -6 & 4 \end{bmatrix}$

Setelah mendapatkan invers matriks, maka system persamaan linier diatas dielesaikan dengan persamaan matriks sebagai berikut :

$$\begin{bmatrix} 2 & 2 & 3 \\ -1 & 1 & 3 \\ -2 & 1 & 2 \end{bmatrix} . \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ -3 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{-7}\begin{bmatrix} -1 & -1 & 3 \\ -4 & 10 & -9 \\ 1 & -6 & 4 \end{bmatrix} . \begin{bmatrix} 4 \\ 1 \\ -3 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{-7}\begin{bmatrix} -14 \\ 21 \\ -14 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 2 \end{bmatrix}$$

Jadi diperoleh nilai x = 2, y = –3 dan z = 2

06. Tentukanlah himpunan penyelesaian sistem persamaan linier

$$\left.\begin{aligned} 2x - 2y - z &= 3 \\ 3x + y + 2z &= 7 \\ 2x + 3y + 4z &= 8 \end{aligned}\right\}$$ dengan menggunakan metoda inver matriks

Jawab

Matriks koefisien untuk sistem persamaan linier diataas adalah :

$$A = \begin{bmatrix} 2 & -2 & -1 \\ 3 & 1 & 2 \\ 2 & 3 & 4 \end{bmatrix}$$ sehingga

Pertama akan ditentukan minor matriks, yaitu

$$M_{11} = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = (1)(4) - (2)(3) = 4 - 6 = -2$$

$$M_{12} = \begin{vmatrix} 3 & 2 \\ 2 & 4 \end{vmatrix} = (3)(4) - (2)(2) = 12 - 4 = 8$$

$$M_{13} = \begin{vmatrix} 3 & 1 \\ 2 & 3 \end{vmatrix} = (3)(3) - (1)(2) = 9 - 2 = 7$$

$$M_{21} = \begin{vmatrix} -2 & -1 \\ 3 & 4 \end{vmatrix} = (-2)(4) - (-1)(3) = -8 + 3 = -5$$

$$M_{22} = \begin{vmatrix} 2 & -1 \\ 2 & 4 \end{vmatrix} = (2)(4) - (-1)(2) = 8 + 2 = 10$$

$$M_{23} = \begin{vmatrix} 2 & -2 \\ 2 & 3 \end{vmatrix} = (2)(3) - (-2)(2) = 6 + 4 = 10$$

$$M_{31} = \begin{vmatrix} -2 & -1 \\ 1 & 2 \end{vmatrix} = (-2)(2) - (-1)(1) = -4 + 1 = -3$$

$$M_{32} = \begin{vmatrix} 2 & -1 \\ 3 & 2 \end{vmatrix} = (2)(2) - (-1)(3) = 4 + 3 = 7$$

$$M_{33} = \begin{vmatrix} 2 & -2 \\ 3 & 1 \end{vmatrix} = (2)(1) - (-2)(3) = 2 + 6 = 8$$

Kemudian menentukan kofaktor matriks

$$C_{11} = (-1)^{1+1}\ M_{11} = (1)(-2) = -2$$

$$C_{12} = (-1)^{1+2}\ M_{12} = (-1)(8) = -8$$

$$C_{13} = (-1)^{1+3}\ M_{13} = (1)(7) = 7$$

$C_{21} = (-1)^{2+1}\ M_{21} = (-1)(-5) = 5$

$C_{22} = (-1)^{2+2}\ M_{22} = (1)(10) = 10$

$C_{23} = (-1)^{2+3}\ M_{23} = (-1)(10) = -10$

$C_{31} = (-1)^{3+1}\ M_{31} = (1)(-3) = -3$

$C_{32} = (-1)^{3+2}\ M_{32} = (-1)(7) = -7$

$C_{33} = (-1)^{3+3}\ M_{33} = (1)(8) = 8$

Matriks kofaktornya : $C = \begin{bmatrix} -2 & -8 & 7 \\ 5 & 10 & -10 \\ -3 & -7 & 8 \end{bmatrix}$

Setelah itu menentukan Determinan matriks menggunakan ekspansi baris pertama

det(A) = $a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} = (-2)(2) + (-8)(-2) + (7)(-1) = 5$

Berikutnya menentukan Adjoint matriks dari matriks kofaktor

Jika kofaktor $C = \begin{bmatrix} -2 & -8 & 7 \\ 5 & 10 & -10 \\ -3 & -7 & 8 \end{bmatrix}$ maka adjoinnya adj(A) = $\begin{bmatrix} -2 & 5 & -3 \\ -8 & 10 & -7 \\ 7 & -10 & 8 \end{bmatrix}$

Akhirnya dimasukkan kedalam rumus invers matriks:

$A^{-1} = \dfrac{1}{\det(A)}$ adj(A) diperoleh $A^{-1} = \dfrac{1}{5}\begin{bmatrix} -2 & 5 & -3 \\ -8 & 10 & -7 \\ 7 & -10 & 8 \end{bmatrix}$

Setelah mendapatkan invers matriks, maka system persamaan linier diatas dielesaikan dengan persamaan matriks sebagai berikut :

$$\begin{bmatrix} 2 & -2 & -1 \\ 3 & 1 & 2 \\ 2 & 3 & 4 \end{bmatrix} . \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \\ 8 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{5}\begin{bmatrix} -2 & 5 & -3 \\ -8 & 10 & -7 \\ 7 & -10 & 8 \end{bmatrix} . \begin{bmatrix} 3 \\ 7 \\ 8 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 5 \\ -10 \\ 15 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$$

Jadi diperoleh nilai x = 1, y = –2 dan z = 3