M A T R I K S

E. Menyelesaikan Persamaan Matriks

Salah satu diantara penggunaan invers matriks adalah untuk menyelesaikan persamaan matriks. Ada dua macam rumus dasar menyelesaikan persamaan matriks, yaitu :

(1) Jika A x B = C maka B = A^{-1} x C

(2) Jika A x B = C maka A = C x B^{-1}

Bukti :

(1) Jika A x B = C maka $A^{-1} \times A \times B = A^{-1} \times C$

$$I \times B = A^{-1} \times C$$

$$B = A^{-1} \times C$$

(2) Jika A x B = C maka $A \times B \times B^{-1} = C \times B^{-1}$

$$A \times I = C \times B^{-1}$$

$$A = C \times B^{-1}$$

Untuk lebih memahami rumus diatas, ikutilah contoh soal berikut ini :

01. Diketahui matriks A = $\begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}$ dan C = $\begin{bmatrix} 3 & 1 \\ -2 & -5 \end{bmatrix}$ maka tentukanlah matriks B jika B x A = C

Jawab

B x A = C

$$B = C \times A^{-1}$$

$$B = \begin{bmatrix} 3 & 1 \\ -2 & -5 \end{bmatrix} \times \frac{1}{8-9}\begin{bmatrix} 4 & -3 \\ -3 & 2 \end{bmatrix}$$

$$B = \begin{bmatrix} 3 & 1 \\ -2 & -5 \end{bmatrix} \times -1\begin{bmatrix} 4 & -3 \\ -3 & 2 \end{bmatrix}$$

$$B = \begin{bmatrix} 3 & 1 \\ -2 & -5 \end{bmatrix} \times \begin{bmatrix} -4 & 3 \\ 3 & -2 \end{bmatrix}$$

$$B = \begin{bmatrix} -12+3 & 9-2 \\ 8-15 & -6+10 \end{bmatrix}$$

$$B = \begin{bmatrix} -9 & 7 \\ -7 & 4 \end{bmatrix}$$

02. Diketahui $A = \begin{bmatrix} 2 & -1 \\ 4 & -3 \end{bmatrix}$, $C = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}$ dan $D = \begin{bmatrix} 0 & 4 \\ -2 & 2 \end{bmatrix}$ maka tentukan matriks B jika A x C x B = D

Jawab

A x C x B = D

$C \times B = A^{-1} \times D$

$B = C^{-1} \times A^{-1} \times D$

$$B = \frac{1}{3-4}\begin{bmatrix} 1 & -2 \\ -2 & 3 \end{bmatrix} \times \frac{1}{-6+4}\begin{bmatrix} -3 & 1 \\ -4 & 2 \end{bmatrix} \times \begin{bmatrix} 0 & 4 \\ -2 & 2 \end{bmatrix}$$

$$B = -1\begin{bmatrix} 1 & -2 \\ -2 & 3 \end{bmatrix} \times -\frac{1}{2}\begin{bmatrix} -3 & 1 \\ -4 & 2 \end{bmatrix} \times \begin{bmatrix} 0 & 4 \\ -2 & 2 \end{bmatrix}$$

$$B = \frac{1}{2}\begin{bmatrix} 1 & -2 \\ -2 & 3 \end{bmatrix} \times \begin{bmatrix} -3 & 1 \\ -4 & 2 \end{bmatrix} \times \begin{bmatrix} 0 & 4 \\ -2 & 2 \end{bmatrix}$$

$$B = \frac{1}{2}\begin{bmatrix} -3+8 & 1-4 \\ 6-12 & -2+6 \end{bmatrix} \times \begin{bmatrix} 0 & 4 \\ -2 & 2 \end{bmatrix}$$

$$B = \frac{1}{2}\begin{bmatrix} 5 & -3 \\ -6 & 4 \end{bmatrix} \times \begin{bmatrix} 0 & 4 \\ -2 & 2 \end{bmatrix}$$

$$B = \frac{1}{2}\begin{bmatrix} 0+6 & 20-6 \\ 0-8 & -24+8 \end{bmatrix}$$

$$B = \frac{1}{2}\begin{bmatrix} 6 & 14 \\ -8 & -16 \end{bmatrix}$$

$$B = \begin{bmatrix} 3 & 7 \\ -4 & -8 \end{bmatrix}$$

03 Diketahui $B = \begin{bmatrix} -2 & 3 \\ 0 & 4 \end{bmatrix}$ dan $C = \begin{bmatrix} 1 & -4 \\ 2 & 2 \end{bmatrix}$. Jika $(A^{-1}.B)^{-1} = C$ maka matriks A adalah …

Jawab

$(A^{-1}.B)^{-1} = C$

$B^{-1} \times (A^{-1})^{-1} = C$

$B^{-1} \times A = C$

$B \times B^{-1} \times A = B \times C$

$I \times A = B \times C$

$A = B \times C$

$$A = \begin{bmatrix} -2 & 3 \\ 0 & 4 \end{bmatrix} \times \begin{bmatrix} 1 & -4 \\ 2 & 2 \end{bmatrix}$$

$$A = \begin{bmatrix} -2+6 & 8+6 \\ 0+8 & 0+8 \end{bmatrix}$$

$$A = \begin{bmatrix} 4 & 14 \\ 8 & 8 \end{bmatrix}$$

04. Jika $A = \begin{bmatrix} 6 & 4 \\ 2 & 8 \end{bmatrix}$, $C = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$ dan $D = \begin{bmatrix} -2 & 0 \\ 1 & 4 \end{bmatrix}$, serta $(A^{-1} x . C)^{-1} (A^{-1} x\ B) = D$ maka tentukanlah matriks B

Jawab

$$(A^{-1} x . C)^{-1} (A^{-1} x\ B) = D$$

$$C^{-1} \ (A^{-1})^{-1} A^{-1} \ B = D$$

$$C^{-1} \ A \ A^{-1} \ B = D$$

$$C^{-1} \ I \ B = D$$

$$C^{-1} \ B = D$$

$$B = C \ x \ D$$

$$B = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix} \ x \ \begin{bmatrix} -2 & 0 \\ 1 & 4 \end{bmatrix}$$

$$B = \begin{bmatrix} -4+1 & 0+4 \\ -10+3 & 0+12 \end{bmatrix}$$

$$B = \begin{bmatrix} -3 & 4 \\ -7 & 12 \end{bmatrix}$$

Kegunaan lain dari invers matriks adalah untuk menentukan penyelesaian sistim persamaan linier. Tentu saja teknik penyelesaiannya dengan aturan persamaan matriks, yaitu :

Jika $\begin{cases} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{cases}$ maka $\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{ad - bc} \begin{bmatrix} b_2 & -b_1 \\ -a_2 & a_1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Selain dengan persamaan matriks, teknik menyelesaikan sistem persamaan linier juga dapat dilakukan dengan determinan matriks. Aturan dengan cara ini adalah :

Jika matriks $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ maka $\det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$ = ad – bc. sehingga

Jika $\begin{cases} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{cases}$ maka $D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - b_1a_2$

$$D_x = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} = c_1 b_2 - b_1 c_2$$

$$D_y = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} = a_1 c_2 - c_1 a_2$$

Maka $x = \frac{D_x}{D}$ dan $y = \frac{D_y}{D}$

Untuk lebih jelanya, ikutolah contoh soal berikut ini:

05. Tentukan himpunan penyelesaian sistem persamaan 2x – 3y = 8 dan x + 2y = –3 dengan metoda:

(a) Invers matriks (b) Determinan

Jawab

2x – 3y = 8

x + 2y = –3 maka $\begin{bmatrix} 2 & -3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ -3 \end{bmatrix}$

(a) Dengan metoda invers matriks diperoleh

$$\begin{bmatrix} 2 & -3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ -3 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{4-(-3)}\begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 8 \\ -3 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 8 \\ -3 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 16-9 \\ -8-6 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 7 \\ -14 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

Jadi x = 1 dan y = –2

(b) Dengan metoda determinan matriks diperoleh

$$D = \begin{vmatrix} 2 & -3 \\ 1 & 2 \end{vmatrix} = (2)(2) - (-3)(1) = 4 + 3 = 7$$

$$D_x = \begin{vmatrix} 8 & -3 \\ -3 & 2 \end{vmatrix} = (8)(2) - (-3)(-3) = 16 - 9 = 7$$

$$D_y = \begin{vmatrix} 2 & 8 \\ 1 & -3 \end{vmatrix} = (2)(-3) - (8)(1) = -6 - 8 = -14$$

Maka $x = \frac{D_x}{D} = \frac{7}{7} = 1$

$y = \frac{D_y}{D} = \frac{-14}{7} = 2$

06. Tentukan himpunan penyelesaian sistem persamaan $y = \frac{1}{2}x + 5$ dan $x + 6 = \frac{2}{3}y$ dengan metoda:

(a) Invers matriks (b) Determinan

Jawab

$y = \frac{1}{2}x + 5 \; \big| (2) \longrightarrow 2y = x + 10 \longrightarrow -x + 2y = 10$

$x + 6 = \frac{2}{3}y \; \big| (3) \longrightarrow 3x + 18 = 2y \longrightarrow 3x - 2y = -18$

$-x + 2y = 10$

$3x - 2y = -18$

Maka

$$D = \begin{vmatrix} -1 & 2 \\ 3 & -2 \end{vmatrix} = (-1)(-2) - (2)(3) = 2 - 6 = -4$$

$$D_x = \begin{vmatrix} 10 & 2 \\ -18 & -2 \end{vmatrix} = (10)(-2) - (2)(-18) = -20 - (-36) = 16$$

$$D_y = \begin{vmatrix} -1 & 10 \\ 3 & -18 \end{vmatrix} = (-1)(-18) - (10)(3) = 18 - 30 = -12$$

Maka $x = \frac{D_x}{D} = \frac{16}{-4} = -4$

$y = \frac{D_y}{D} = \frac{-12}{-4} = 3$

(3) Sistem persamaan linier tiga variabel

Jika $\begin{cases} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \\ a_3x + b_3y + c_3z = d_3 \end{cases}$ diperoleh nilai determinan :

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1.b_2.c_3 + b_1.c_2.a_3 + c_1.a_2.b_3 - c_1.b_2.a_3 - a_1.c_2.b_3 - b_1.a_2.c_3$$

$$D_x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix} = d_1.b_2.c_3 + b_1.c_2.d_3 + c_1.d_2.b_3 - c_1.b_2.d_3 - d_1.c_2.b_3 - b_1.d_2.c_3$$

$$D_y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} = a_1.d_2.c_3 + d_1.c_2.a_3 + c_1.a_2.d_3 - c_1.d_2.a_3 - a_1.c_2.d_3 - d_1.a_2.c_3$$

$$D_z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix} = a_1.b_2.d_3 + b_1.d_2.a_3 + d_1.a_2.b_3 - d_1.b_2.a_3 - a_1.d_2.b_3 - b_1.a_2.d_3$$

Sehingga nilai $x = \frac{D_x}{D}$, $y = \frac{D_y}{D}$ dan $z = \frac{D_z}{D}$

Untuk lebih jelanya, ikutolah contoh soal berikut ini:

07. Tentukanlah himpunan penyelesaian sistem persamaan linier

$$\left.\begin{array}{l} x + 2y + z = 2 \\ x - y - 2z = -1 \\ x + y - z = 3 \end{array}\right\}$$ dengan menggunakan metoda determinan

Jawab

$$D = \begin{vmatrix} 1 & 2 & 1 \\ 1 & -1 & -2 \\ 1 & 1 & -1 \end{vmatrix} = \left|\begin{array}{ccc:cc} 1 & 2 & 1 & 1 & 2 \\ 1 & -1 & -2 & 1 & -1 \\ 1 & 1 & -1 & 1 & 1 \end{array}\right|$$

D = (1)(–1)(–1) + (2)(–2)(1) + (1)(1)(1) – (1)(–1)(1) – (1)(–2)(1) – (2)(1)(–1)

D = 1 – 4 + 1 + 1 + 2 + 2

D = 3

$$D_x = \begin{vmatrix} 2 & 2 & 1 \\ -1 & -1 & -2 \\ 3 & 1 & -1 \end{vmatrix} = \left|\begin{array}{ccc:cc} 2 & 2 & 1 & 2 & 2 \\ -1 & -1 & -2 & -1 & -1 \\ 3 & 1 & -1 & 3 & 1 \end{array}\right|$$

D_x = (2)(–1)(–1) + (2)(–2)(3) + (1)(–1)(1) – (1)(–1)(3) – (2)(–2)(1) – (2)(–1)(–1)

D_x = 2 – 12 – 1 + 3 + 4 – 2

D_x = –6

$$D_y = \begin{vmatrix} 1 & 2 & 1 \\ 1 & -1 & -2 \\ 1 & 3 & -1 \end{vmatrix} = \left|\begin{array}{ccc:cc} 1 & 2 & 1 & 1 & 2 \\ 1 & -1 & -2 & 1 & -1 \\ 1 & 3 & -1 & 1 & 3 \end{array}\right|$$

D_y = (1)(–1)(–1) + (2)(–2)(1) + (1)(1)(3) – (1)(–1)(1) – (1)(–2)(3) – (2)(1)(–1)

D_y = 1 – 4 + 3 + 1 + 6 + 2

D_y = 9

$$D_z = \begin{vmatrix} 1 & 2 & 2 \\ 1 & -1 & -1 \\ 1 & 1 & 3 \end{vmatrix} = \left|\begin{array}{ccc:cc} 1 & 2 & 2 & 1 & 2 \\ 1 & -1 & -1 & 1 & -1 \\ 1 & 1 & 3 & 1 & 1 \end{array}\right|$$

D_z = (1)(–1)(3) + (2)(–1)(1) + (2)(1)(1) – (2)(–1)(1) – (1)(–1)(1) – (2)(1)(3)

D_z = –3 – 2 + 2 + 2 + 1 – 6

D_z = –6

Jadi $x = \frac{D_x}{D} = \frac{-6}{3} = -2$

$y = \frac{D_y}{D} = \frac{9}{3} = 3$

$z = \frac{D_z}{D} = \frac{-6}{3} = -2$

08. Tentukanlah himpunan penyelesaian sistem persamaan linier

$$\left.\begin{aligned} x - 2y &= -3 \\ y + z &= 1 \\ 2x + z &= 1 \end{aligned}\right\}$$ dengan menggunakan metoda determinan

Jawab

$$D = \begin{vmatrix} 1 & -2 & 0 \\ 0 & 1 & 1 \\ 2 & 0 & 1 \end{vmatrix} = \left|\begin{array}{ccc:cc} 1 & -2 & 0 & 1 & -2 \\ 0 & 1 & 1 & 0 & 1 \\ 2 & 0 & 1 & 2 & 0 \end{array}\right|$$

D = (1)(1)(1) + (–2)(1)(2) + (0)(0)(0) – (0)(1)(2) – (1)(1)(0) – (–2)(0)(1)

D = (1) + (–4) + (0) – (0) – (0) – (0)

D = 1 – 4 + 0 + 0 + 0 + 0

D = –3

$$D_x = \begin{vmatrix} -3 & -2 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix} = \left|\begin{array}{ccc:cc} -3 & -2 & 0 & -3 & -2 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 \end{array}\right|$$

D_x = (–3)(1)(1) + (–2)(1)(1) + (0)(1)(0) – (0)(1)(1) – (–3)(1)(0) – (–2)(1)(1)

D_x = (–3) + (–2) + (0) – (0) – (0) – (–2)

D_x = –3 – 2 + 0 – 0 – 0 + 2

D_x = –3

$$D_y = \begin{vmatrix} 1 & -3 & 0 \\ 0 & 1 & 1 \\ 2 & 1 & 1 \end{vmatrix} = \left|\begin{array}{ccc:cc} 1 & -3 & 0 & 1 & -3 \\ 0 & 1 & 1 & 0 & 1 \\ 2 & 1 & 1 & 2 & 1 \end{array}\right|$$

D_y = (1)(1)(1) + (–3)(1)(2) + (0)(0)(1) – (0)(1)(2) – (1)(1)(1) – (–3)(0)(1)

D_y = (1) + (–6) + (0) – (0) – (1) – (0)

D_y = 1 – 6 + 0 + 0 – 1 – 0

D_y = –6

$$D_z = \begin{vmatrix} 1 & -2 & -3 \\ 0 & 1 & 1 \\ 2 & 0 & 1 \end{vmatrix} = \left|\begin{array}{ccc:cc} 1 & -2 & -3 & 1 & -2 \\ 0 & 1 & 1 & 0 & 1 \\ 2 & 0 & 1 & 2 & 0 \end{array}\right|$$

D_z = (1)(1)(1) + (–2)(1)(2) + (–3)(0)(0) – (–3)(1)(2) – (1)(1)(0) – (–2)(0)(1)

D_z = (1) + (–4) + (0) – (–6) – (0) – (0)

D_z = 1 – 4 + 0 + 6 – 0 – 0

D_z = 3

Jadi $x = \frac{D_x}{D} = \frac{-3}{-3} = 1$

$y = \frac{D_y}{D} = \frac{-6}{-3} = 2$

$z = \frac{D_z}{D} = \frac{3}{-3} = -1$