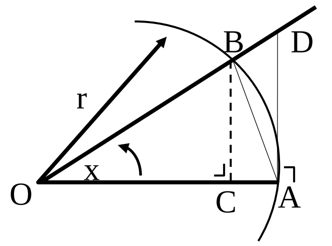
LIMIT FUNGSI

B. Limit FungsiTrigonometri



Misalkan x dalam radian, dan $0 < x < \frac{\pi}{2}$, maka

BC = r sin x dan AD = r tan x.

Untuk mencari luas juring AOB

$$\frac{\text{Luas Juring OAB}}{\text{Luas Seluruh Lingkaran}} = \frac{x}{2\pi}$$

$$\frac{\text{Luas Juring OAB}}{\pi . r^2} = \frac{x}{2\pi}$$

Sehingga luas juring AOB = $\frac{x}{2\pi}.\pi r^2 = \frac{1}{2} r^2 x$

Dari bangun di atas diperoleh :

luas Δ AOB <luas juring AOB <luas Δ AOD

$\frac{1}{2}$. OA .BC < $\frac{1}{2} r^2 x$ < $\frac{1}{2}$ .OA . AD

$\frac{1}{2}$. r . r sin x < $\frac{1}{2} r^2$ x < $\frac{1}{2}$. r . r tan x

$\frac{1}{2} r^2$ sin x < $\frac{1}{2} r^2$ x < $\frac{1}{2} r^2$ tan x

sin x < x < tan x ………………………………………………………………………….. (i)

Dari (i) diperoleh :

$$1 < \frac{x}{\sin x} < \frac{1}{\cos x}$$

$$\lim_{x\to 0} 1 \le \lim_{x\to 0} \frac{x}{\sin x} \le \lim_{x\to 0} \frac{1}{\cos x}$$

$1 \le \lim_{x\to 0} \frac{x}{\sin x} \le \frac{1}{1} = 1$. Jadi $\lim_{x\to 0} \frac{x}{\sin x} = 1$ ……………………………… (1)

Dari sini dapat dikembangkan :

$\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{1}{\frac{x}{\sin x}} = \frac{1}{1} = 1$ jadi $\lim_{x\to 0} \frac{\sin x}{x} = 1$ ……………………………..…….. (2)

Dan untuk $\lim_{x\to 0} \frac{\tan x}{x} = \lim_{x\to 0} \frac{\sin x}{x.\cos x}$

$$= \lim_{x\to 0} \frac{\sin x}{x} . \frac{1}{\cos x}$$

$$= \lim_{x\to 0} \frac{\sin x}{x} . \lim_{x\to 0} \frac{1}{\cos x} = 1 \quad \text{jadi} \quad \lim_{x\to 0} \frac{\tan x}{x} = 1 \quad \ldots\ldots\ldots\ldots\ldots (3)$$

Demikian juga dengan mudah dapat ditunjukkan bahwa $\lim\limits_{x\to 0} \frac{x}{\tan x} = 1$ ………… (4)

Dari uraian diatas diperoleh rumus dasar limit fungsi trigonometri, yaitu :

(1) $\underset{x\to 0}{\text{Limit}} \frac{\sin x}{x} = 1$ (2) $\underset{x\to 0}{\text{Limit}} \frac{x}{\sin x} = 1$

(3) $\underset{x\to 0}{\text{Limit}} \frac{\tan x}{x} = 1$ (4) $\underset{x\to 0}{\text{Limit}} \frac{x}{\tan x} = 1$

Dari rumus dasar diatas dapat dikembangkan rumus-rumus sebagai berikut:

$$\underset{x\to 0}{\text{Limit}} \frac{\sin ax}{bx} = \underset{x\to 0}{\text{Limit}} \frac{\sin ax}{bx} \times \frac{a}{a} = \underset{x\to 0}{\text{Limit}} \frac{\sin ax}{ax} \times \frac{a}{b} = \frac{a}{b}$$

$$\underset{x\to 0}{\text{Limit}} \frac{bx}{\sin ax} = \underset{x\to 0}{\text{Limit}} \frac{bx}{\sin ax} \times \frac{a}{a} = \underset{x\to 0}{\text{Limit}} \frac{ax}{\sin ax} \times \frac{b}{a} = \frac{b}{a}$$

$$\underset{x\to 0}{\text{Limit}} \frac{\tan ax}{bx} = \underset{x\to 0}{\text{Limit}} \frac{\tan ax}{bx} \times \frac{a}{a} = \underset{x\to 0}{\text{Limit}} \frac{\tan ax}{ax} \times \frac{a}{b} = \frac{a}{b}$$

$$\underset{x\to 0}{\text{Limit}} \frac{bx}{\tan ax} = \underset{x\to 0}{\text{Limit}} \frac{bx}{\tan ax} \times \frac{a}{a} = \underset{x\to 0}{\text{Limit}} \frac{ax}{\tan ax} \times \frac{b}{a} = \frac{b}{a}$$

$$\underset{x\to 0}{\text{Limit}} \frac{\sin ax}{\tan bx} = \underset{x\to 0}{\text{Limit}} \frac{\sin ax}{\tan bx} \times \frac{ax}{ax} \times \frac{bx}{bx} = \underset{x\to 0}{\text{Limit}} \frac{\sin ax}{ax} \times \frac{bx}{\tan bx} \times \frac{ax}{bx} = \frac{a}{b}$$

Atau dapat disimpulkan :

(1) $\underset{x\to 0}{\text{Limit}} \frac{\sin ax}{bx} = \frac{a}{b}$ dan $\underset{x\to 0}{\text{Limit}} \frac{bx}{\sin ax} = \frac{b}{a}$

(2) $\underset{x\to 0}{\text{Limit}} \frac{\tan ax}{bx} = \frac{a}{b}$ dan $\underset{x\to 0}{\text{Limit}} \frac{bx}{\tan ax} = \frac{b}{a}$

(3) $\underset{x\to 0}{\text{Limit}} \frac{\sin ax}{\tan bx} = \frac{a}{b}$ dan $\underset{x\to 0}{\text{Limit}} \frac{\tan bx}{\sin ax} = \frac{b}{a}$

Untuk lebih jelasnya ikutilah contoh soal berikut ini :

01. Tentukanlah hasil setiap limit fungsi trigonometri berikutini

(a) $\underset{x\to 0}{\text{Limit}} \left[\frac{\sin 4x}{3x} - \frac{\tan 2x}{\sin 6x} + \frac{8x}{\tan 2x}\right]$ (b) $\underset{x\to 0}{\text{Limit}} \frac{\sin^2 6x}{3x.\tan 4x}$

(c) $\underset{x\to 0}{\text{Limit}} \frac{2\tan^2 3x.\sin 2x}{4x^2.\sin 6x}$ (d) $\underset{x\to 0}{\text{Limit}} \frac{6.\sin^3 2x}{\sin 4x.\sin 3x}$

(e) $\underset{x\to 0}{\text{Limit}} \left[\frac{\sin 2x + 4x}{\sin 3x + \tan x}\right]$ (f) $\underset{x\to 0}{\text{Limit}} \left[\frac{6x^2 + \sin^2 3x}{\tan^2 2x - x^2}\right]$

Jawab

(a) $\lim_{x\to 0}\left[\frac{\sin 4x}{3x}-\frac{\tan 2x}{\sin 6x}+\frac{8x}{\tan 2x}\right] = \frac{4}{3} - \frac{2}{6} + \frac{8}{2} = \frac{4}{3} - \frac{1}{3} + 4 = 5$

(b) $\lim_{x\to 0}\frac{\sin^2 6x}{3x.\tan 4x} = \lim_{x\to 0}\frac{\sin 6x}{3x}.\frac{\sin 6x}{\tan 4x} = \left(\frac{6}{3}\right)\left(\frac{6}{4}\right) = \frac{36}{12} = 3$

(c) $\lim_{x\to 0}\frac{2\tan^2 3x.\sin 2x}{4x^2.\sin 6x} = \lim_{x\to 0} 2\left(\frac{\tan 3x}{4x}\right)\left(\frac{\tan 3x}{x}\right)\left(\frac{\sin 2x}{\sin 6x}\right)$

$= 2\left(\frac{3}{4}\right)\left(\frac{3}{1}\right)\left(\frac{2}{6}\right)$

$= 3/2$

(d) $\lim_{x\to 0}\frac{6.\sin^3 2x}{\sin 4x.\sin 3x} = \lim_{x\to 0} 6\left(\frac{\sin 2x}{\sin 4x}\right)\left(\frac{\sin 2x}{\sin 3x}\right)\sin 2x$

$= 6\left(\frac{2}{4}\right)\left(\frac{2}{3}\right)\sin 2(0)$

$= \left(\frac{24}{12}\right)0$

$= 0$

(e) $\lim_{x\to 0}\left[\frac{\sin 2x + 4x}{\sin 3x + \tan x}\right] = \lim_{x\to 0}\left[\frac{\frac{\sin 2x}{x}+\frac{4x}{x}}{\frac{\sin 3x}{x}+\frac{\tan x}{x}}\right]$

$= \left[\frac{\frac{2}{1}+\frac{4}{1}}{\frac{3}{1}+\frac{1}{1}}\right]$

$= \left(\frac{2+4}{3+1}\right)$

$= 3/2$

(e) $\lim_{x\to 0}\left[\frac{6x^2+\sin^2 3x}{\tan^2 2x - x^2}\right] = \lim_{x\to 0}\left[\frac{\frac{6x^2}{x^2}+\frac{\sin^2 3x}{x^2}}{\frac{\tan^2 2x}{x^2}-\frac{x^2}{x^2}}\right]$

$= \left[\frac{\frac{6}{1}+\frac{9}{1}}{\frac{4}{1}-\frac{1}{1}}\right]$

$= \left(\frac{6+9}{4-1}\right)$

$= 5$

Menyesuaikan dengan rumus limit fungsi trigonometri diatas, jika p = x – a maka untuk nilai x mendekati a diperoleh nilai p mendekati 0, sehingga :

$$\underset{x \to a}{\text{Limit}} \frac{\sin a(x-a)}{b(x-a)} = \underset{p \to 0}{\text{Limit}} \frac{\sin ap}{bp} = \frac{a}{b}$$

$$\underset{x \to a}{\text{Limit}} \frac{\tan a(x-a)}{b(x-a)} = \underset{p \to 0}{\text{Limit}} \frac{\tan ap}{bp} = \frac{a}{b}$$

Dan juga berlaku untuk rumus-rumus limit fungsi trigonometri yang lain.
Untuk lebih jelasnya ikutilah contoh soal berikut ini :

02. Tentukanlah hasil setiap limit fungsi trigonometri berikut ini

(a) $\underset{x \to 2}{\text{Limit}} \frac{3\sin(x-2)}{(4x-8)}$

(b) $\underset{x \to 3}{\text{Limit}} \frac{6\tan^2(2x-6)}{(3x-9)^2}$

(c) $\underset{x \to 4}{\text{Limit}} \left[\frac{\sin(2x-8)}{\tan(x-4)+(3x-12)}\right]$

(d) $\underset{x \to 1}{\text{Limit}} \frac{\tan(2x^2-6x+4)}{3x^2-9x+6}$

(e) $\underset{x \to 2}{\text{Limit}} \frac{\sin(3x-6)}{x^2+2x-8}$

Jawab

(a) $\underset{x \to 2}{\text{Limit}} \frac{3\sin(x-2)}{(4x-8)} = \underset{x \to 2}{\text{Limit}} \frac{3\sin(x-2)}{4(x-2)} = 3\left(\frac{1}{4}\right) = \frac{3}{4}$

(b) $\underset{x \to 3}{\text{Limit}} \frac{6\tan^2(2x-6)}{(3x-9)^2} = \underset{x \to 3}{\text{Limit}}\ 6\left(\frac{\tan(2x-6)}{3x-9}\right)^2$

$= \underset{x \to 3}{\text{Limit}}\ 6\left(\frac{\tan 2(x-3)}{3(x-3)}\right)^2$

$= 6.\left(\frac{2}{3}\right)^2$

= 8/3

(c) $\underset{x \to 4}{\text{Limit}} \left[\frac{\sin(2x-8)}{\tan(x-4)+(3x-12)}\right] = \underset{x \to 4}{\text{Limit}} \left[\frac{\sin 2(x-4)}{\tan(x-4)+3(x-4)}\right]$

$= \underset{x \to 4}{\text{Limit}} \left[\frac{\frac{\sin 2(x-4)}{(x-4)}}{\frac{\tan(x-4)}{(x-4)} + \frac{3(x-4)}{(x-4)}}\right]$

$= \left(\frac{2}{1+3}\right)$

= 1/2

(d) $\lim\limits_{x \to 1} \dfrac{\tan(2x^2 - 6x + 4)}{3x^2 - 9x + 6} = \lim\limits_{x \to 1} \dfrac{\tan 2(x^2 - 3x + 2)}{3(x^2 - 3x + 2)}$

$= \lim\limits_{x \to 1} \dfrac{\tan 2(x + 3)(x - 1)}{3(x + 3)(x - 1)}$

$= \dfrac{2}{3}$

(e) $\lim\limits_{x \to 2} \dfrac{\sin(3x - 6)}{x^2 + 2x - 8} = \lim\limits_{x \to 2} \dfrac{\sin 3(x - 2)}{(x + 4)(x - 2)}$

$= \lim\limits_{x \to 2} \dfrac{1}{(x + 4)} \cdot \dfrac{\sin 3(x - 2)}{(x - 2)}$

$= \left(\dfrac{1}{2 + 4}\right)\left(\dfrac{3}{1}\right)$

$= \dfrac{3}{6}$

$= \dfrac{1}{2}$

Disamping rumus pengembangan di atas sering pula digunakan rumus rumus trigonometri lainnya yang telah dipelajari pada bab sebelumnya, yakni

(1) $1 - \cos 2\alpha = 2 \sin^2 \alpha$

(2) $\cos 2\alpha - 1 = -2 \sin^2 \alpha$

(3) $1 - \cos^2 \alpha = \sin^2 \alpha$

(4) $\text{ctg}\,\alpha = \dfrac{1}{\tan \alpha}$

(5) $\cos A - \cos B = -2\sin \frac{1}{2}(A + B).\sin \frac{1}{2}(A - B)$

Untuk lebih jelasnya pemakaian rumus-rumus di atas, ikutilah contoh soal berikut ini

03. Tentukanlah hasil setiap limit fungsi trigonometri berikut ini

(a) $\lim\limits_{x \to 0} \dfrac{1 - \cos 6x}{\cos 8x - 1}$

(b) $\lim\limits_{x \to 0} \dfrac{3\cos 4x - 3}{2.\sin^2 3x}$

(c) $\lim\limits_{x \to 0} \dfrac{4 - 4\cos 2x}{1 - \cos^2 3x}$

(d) $\lim\limits_{x \to 0} \dfrac{2 - 2\cos^2 6x}{3\cos^2 2x - 3}$

(e) $\lim\limits_{x \to 0} \dfrac{\cos 5x - \cos 3x}{1 - \cos 8x}$

(f) $\lim\limits_{x \to 0} \dfrac{3\cos 6x - 3\cos 2x}{1 - \cos^2 2x}$

Jawab

(a) $\underset{x\to 0}{\text{Limit}} \dfrac{1-\cos 6x}{\cos 8x - 1} = \underset{x\to 0}{\text{Limit}} \dfrac{1-\cos 2(3x)}{\cos 2(4x)-1}$

$= \underset{x\to 0}{\text{Limit}} \dfrac{2.\sin^2 3x}{-2.\sin^2 4x}$

$= \underset{x\to 0}{\text{Limit}} -\left(\dfrac{.\sin 3x}{.\sin 4x}\right)^2$

$= -\left(\dfrac{.3}{.4}\right)^2$

$= -\dfrac{.9}{.16}$

(b) $\underset{x\to 0}{\text{Limit}} \dfrac{3\cos 4x - 3}{2.\sin^2 3x} = \underset{x\to 0}{\text{Limit}} \dfrac{3(\cos 2(2x)-1)}{2.\sin^2 3x}$

$= \underset{x\to 0}{\text{Limit}} \dfrac{-3.\sin^2 2x}{2.\sin^2 3x}$

$= \underset{x\to 0}{\text{Limit}} -\dfrac{3}{2}\left(\dfrac{.\sin 2x}{.\sin 3x}\right)^2$

$= -\dfrac{3}{2}\left(\dfrac{.2}{.3}\right)^2$

$= -\dfrac{.2}{.3}$

(c) $\underset{x\to 0}{\text{Limit}} \dfrac{4-4\cos 2x}{1-\cos^2 3x} = \underset{x\to 0}{\text{Limit}} \dfrac{4(1-\cos 2x)}{1-\cos^2 3x}$

$= \underset{x\to 0}{\text{Limit}} \dfrac{4.(2\sin^2 x)}{\sin^2 3x}$

$= \underset{x\to 0}{\text{Limit}}\ 8\left(\dfrac{.\sin x}{.\sin 3x}\right)^2$

$= 8\left(\dfrac{1}{.3}\right)^2$

$= \dfrac{8}{9}$

(d) $\underset{x\to 0}{\text{Limit}} \dfrac{2-2\cos^2 6x}{3\cos^2 2x - 3} = \underset{x\to 0}{\text{Limit}} \dfrac{2(1-\cos^2 6x)}{3(\cos^2 2x - 1)}$

$= \underset{x\to 0}{\text{Limit}} \dfrac{4.(\sin^2 6x)}{3.(-\sin^2 2x)}$

$= \underset{x\to 0}{\text{Limit}} -\dfrac{4}{3}\left(\dfrac{\sin 6x}{\sin 2x}\right)^2$

$= -\dfrac{4}{3}\left(\dfrac{6}{2}\right)^2$

$= -\dfrac{4}{3}(9)$

$= -12$

(e) $\underset{x\to 0}{\text{Limit}} \dfrac{\cos 5x - \cos 3x}{1-\cos 8x} = \underset{x\to 0}{\text{Limit}} \dfrac{-2\sin\frac{1}{2}(5x+3x).\sin\frac{1}{2}(5x-3x)}{1-\cos 2(4x)}$

$= \underset{x\to 0}{\text{Limit}} \dfrac{-2\sin 4x.\sin x}{2\sin^2 4x}$

$= \underset{x\to 0}{\text{Limit}} \dfrac{-\sin x}{\sin 4x}$

$= -\dfrac{1}{4}$

(f) $\underset{x\to 0}{\text{Limit}} \dfrac{3\cos 6x - 3\cos 2x}{1-\cos^2 2x} = \underset{x\to 0}{\text{Limit}} \dfrac{3(\cos 6x - \cos 2x)}{1-\cos^2 2x}$

$= \underset{x\to 0}{\text{Limit}} \dfrac{-6\sin\frac{1}{2}(6x+2x).\sin\frac{1}{2}(6x-2x)}{\sin^2(2x)}$

$= \underset{x\to 0}{\text{Limit}} \dfrac{-6\sin 4x.\sin 2x}{\sin 2x.\sin 2x}$

$= \underset{x\to 0}{\text{Limit}} \dfrac{-6\sin 4x}{\sin 2x}$

$= -6\left(\dfrac{4}{2}\right)$

$= -12$

Terdapat pula limit fungsi trigonometri yang penyelesaiannya tidak menggunakan cara-cara seperti diatas. Sebagai contoh akan diberikan pada soal berikut ini :

04. Tentukanlah hasil setiap limit fungsi trigonometri berikut ini

(a) $\underset{x \to \frac{\pi}{2}}{\text{Limit}} \dfrac{3.\sin 4x}{\cos x}$ (b) $\underset{x \to \frac{\pi}{4}}{\text{Limit}} \dfrac{\sin 6x + \sin 2x}{\cos 5x + \cos x}$

Jawab

(a) $\underset{x \to \frac{\pi}{2}}{\text{Limit}} \dfrac{3.\sin 4x}{\cos x} = \underset{x \to \frac{\pi}{2}}{\text{Limit}} \dfrac{3.(2\sin 2x.\cos 2x)}{\cos x}$

$= \underset{x \to \frac{\pi}{2}}{\text{Limit}} \dfrac{6\sin 2x.\cos 2x}{\cos x}$

$= \underset{x \to \frac{\pi}{2}}{\text{Limit}} \dfrac{6.(2\sin x.\cos x).\cos 2x}{\cos x}$

$= \underset{x \to \frac{\pi}{2}}{\text{Limit}}$ 12.sin x . cos 2x

$= 12 \sin\dfrac{\pi}{2}.\cos 2\left(\dfrac{\pi}{2}\right)$

$= 12 \sin\dfrac{\pi}{2}.\cos \pi$

$= 12\,(1)(0)$

$= 0$

(b) $\underset{x \to \frac{\pi}{4}}{\text{Limit}} \dfrac{\sin 6x + \sin 2x}{\cos 5x + \cos x} = \underset{x \to \frac{\pi}{4}}{\text{Limit}} \dfrac{2.\sin 3x.\cos 2x}{2.\cos 4x.\cos 2x}$

$= \underset{x \to \frac{\pi}{4}}{\text{Limit}} \dfrac{\sin 3x}{\cos 4x}$

$= \dfrac{\sin 3\left(\dfrac{\pi}{4}\right)}{\cos 4\left(\dfrac{\pi}{4}\right)}$

$= \dfrac{\sin \dfrac{3\pi}{4}}{\cos \pi}$

$= \dfrac{\dfrac{1}{2}\sqrt{3}}{-1}$

$= -\dfrac{1}{2}\sqrt{3}$