M A T R I K S

F. Invers Perkalian Matriks ordo (3 x 3)

Seperti yang telah diuraikan di atas, bahwa setiap matriks persegi mempunyai identitas perkalian (dilambangkan dengan I) dan invers perkalian, sehingga berlaku :
Jika A^{-1} adalah invers dari matriks A, maka A^{-1} x A = A x A^{-1} = I

Selanjutnya akan dibahas tentang matriks identitas dan invers perkalian matriks persegi ordo (3 x 3).

Matriks identitas perkalian ordo (3 x 3) adalah I = $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ Sedangkan untuk menentukan invers perkalian matriks (3 x 3) dapat dilakukan dengan dua cara, yaitu :

(1) Dengan metoda mereduksi elemen baris.

Untuk menentukan invers matriks dengan metoda ini, dilakukan dengan cara :

Jika A = $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ maka invers matriks A didapat dengan cara mereduksi elemen baris matriks A, sehingga :

$$\left|\begin{array}{ccc|ccc} a & b & c & 1 & 0 & 0 \\ d & e & f & 0 & 1 & 0 \\ g & h & i & 0 & 0 & 1 \end{array}\right| \quad \text{diubah menjadi} \quad \left|\begin{array}{ccc|ccc} 1 & 0 & 0 & p & q & r \\ 0 & 1 & 0 & s & t & u \\ 0 & 0 & 1 & v & w & x \end{array}\right|$$

dalam hal ini $A^{-1} = \begin{bmatrix} p & q & r \\ s & t & u \\ v & w & x \end{bmatrix}$

Terdapat beberapa aturan dalam reduksi elemen baris, yaitu :

(1) Setiap elemen baris dapat dikali (atau dibagi) dengan bilangan real
(2) Setiap elemen baris dapat ditambah (atau dikurang) dengan elemen baris yang lain
(3) Setiap elemen baris dapat ditukar posisi dengan baris lain

Untuk lebih jelasnya ikutilah contoh soal berikut ini :

01. Tentukanlah invers matriks A = $\begin{bmatrix} 1 & -2 & 1 \\ 2 & -3 & 2 \\ 0 & -5 & 1 \end{bmatrix}$

Jawab

$$\left|\begin{array}{ccc|ccc|l} 1 & -2 & 1 & 1 & 0 & 0 & b_1 \times 2 \\ 2 & -3 & 2 & 0 & 1 & 0 & \\ 0 & -5 & 1 & 0 & 0 & 1 & \end{array}\right.$$

$$\left|\begin{array}{ccc|ccc|l} 2 & -4 & 2 & 2 & 0 & 0 & \\ 2 & -3 & 2 & 0 & 1 & 0 & b_2 - b_1 \\ 0 & -5 & 1 & 0 & 0 & 1 & \end{array}\right.$$

$$\left|\begin{array}{ccc|ccc|l} 2 & -4 & 2 & 2 & 0 & 0 & \\ 0 & 1 & 0 & -2 & 1 & 0 & b_2 \times 5 \\ 0 & -5 & 1 & 0 & 0 & 1 & \end{array}\right.$$

$$\left|\begin{array}{ccc|ccc|l} 2 & -4 & 2 & 2 & 0 & 0 & \\ 0 & 5 & 0 & -10 & 5 & 0 & b_2 \times 5 \\ 0 & -5 & 1 & 0 & 0 & 1 & \end{array}\right.$$

$$\left|\begin{array}{ccc|ccc|l} 2 & -4 & 2 & 2 & 0 & 0 & \\ 0 & 5 & 0 & -10 & 5 & 0 & \\ 0 & 0 & 1 & -10 & 5 & 1 & b_3 + b_2 \end{array}\right.$$

$$\left|\begin{array}{ccc|ccc|l} 1 & -2 & 1 & 1 & 0 & 0 & b_1 : 2 \\ 0 & 1 & 0 & -2 & 1 & 0 & b_2 : 5 \\ 0 & 0 & 1 & -10 & 5 & 1 & \end{array}\right.$$

$$\left|\begin{array}{ccc|ccc|l} 1 & -2 & 0 & 11 & -5 & -1 & b_1 - b_3 \\ 0 & 1 & 0 & -2 & 1 & 0 & \\ 0 & 0 & 1 & -10 & 5 & 1 & \end{array}\right.$$

$$\left|\begin{matrix} 1 & -2 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{matrix}\right|\begin{matrix} 11 & -5 & -1 \\ -4 & 2 & 0 \\ -10 & 5 & 1 \end{matrix}\left| \; b_2 \times 2 \right.$$

$$\left|\begin{matrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{matrix}\right|\begin{matrix} 7 & -3 & -1 \\ -4 & 2 & 0 \\ -10 & 5 & 1 \end{matrix}\left| \; b_1 + b_2 \right.$$

$$\left|\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}\right|\begin{matrix} 7 & -3 & -1 \\ -2 & 1 & 0 \\ -10 & 5 & 1 \end{matrix}\left| \; b_2 : 2 \right.$$

maka $A^{-1} = \begin{bmatrix} a & -3 & b \\ c & d & 0 \\ -10 & e & f \end{bmatrix} = \begin{bmatrix} 7 & -3 & -1 \\ -2 & 1 & 0 \\ -10 & 5 & 1 \end{bmatrix}$

02. Tentukanlah invers matriks A = $\begin{bmatrix} -1 & 2 & -3 \\ 2 & 1 & 0 \\ 4 & -2 & 5 \end{bmatrix}$

Jawab

$$\left|\begin{matrix} -1 & 2 & -3 \\ 2 & 1 & 0 \\ 4 & -2 & 5 \end{matrix}\right|\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}\left| \; \begin{matrix} b_1 \times 5 \\ \\ b_3 \times 3 \end{matrix} \right.$$

$$\left|\begin{matrix} -5 & 10 & -15 \\ 2 & 1 & 0 \\ 12 & -6 & 15 \end{matrix}\right|\begin{matrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{matrix}\left| \; b_1 + b_3 \right.$$

$$\left|\begin{matrix} 7 & 4 & 0 \\ 8 & 4 & 0 \\ 12 & -6 & 15 \end{matrix}\right|\begin{matrix} 5 & 0 & 3 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{matrix}\left| \; b_2 \times 4 \right.$$

$$\left|\begin{matrix} -1 & 0 & 0 \\ 8 & 4 & 0 \\ 12 & -6 & 15 \end{matrix}\right| \begin{matrix} 5 & -4 & 3 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{matrix}\right| \quad b_1 - b_2$$

$$\left|\begin{matrix} -4 & 0 & 0 \\ 8 & 4 & 0 \\ 4 & -2 & 5 \end{matrix}\right| \begin{matrix} 20 & -16 & 12 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{matrix}\right| \quad \begin{matrix} b_1 \times 4 \\ \\ b_3 : 3 \end{matrix}$$

$$\left|\begin{matrix} -4 & 0 & 0 \\ 8 & 4 & 0 \\ 0 & -2 & 5 \end{matrix}\right| \begin{matrix} 20 & -16 & 12 \\ 0 & 4 & 0 \\ 20 & -16 & 13 \end{matrix}\right| \quad \begin{matrix} \\ \\ b_3 + b_1 \end{matrix}$$

$$\left|\begin{matrix} -2 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & -2 & 5 \end{matrix}\right| \begin{matrix} 10 & -8 & 6 \\ 0 & 1 & 0 \\ 20 & -16 & 13 \end{matrix}\right| \quad \begin{matrix} b_1 : 2 \\ b_2 : 4 \\ \\ \end{matrix}$$

$$\left|\begin{matrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 5 \end{matrix}\right| \begin{matrix} 10 & -8 & 6 \\ 10 & -7 & 6 \\ 20 & -16 & 13 \end{matrix}\right| \quad b_2 + b_1$$

$$\left|\begin{matrix} -2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & -2 & 5 \end{matrix}\right| \begin{matrix} 10 & -8 & 6 \\ 20 & -14 & 12 \\ 20 & -16 & 13 \end{matrix}\right| \quad b_2 \times 2$$

$$\left|\begin{matrix} -2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{matrix}\right| \begin{matrix} 10 & -8 & 6 \\ 20 & -14 & 12 \\ 40 & -30 & 25 \end{matrix}\right| \quad \begin{matrix} \\ \\ b_3 + b_2 \end{matrix}$$

$$\left|\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}\right| \begin{matrix} -5 & 4 & -3 \\ 10 & -7 & 6 \\ 8 & -6 & 5 \end{matrix}\right| \quad \begin{matrix} b_1 : (-2) \\ b_2 : 2 \\ b_3 : 5 \end{matrix}$$

maka $A^{-1} = \begin{bmatrix} a & 4 & b \\ c & d & 6 \\ 8 & e & f \end{bmatrix} = \begin{bmatrix} -5 & 4 & -3 \\ 10 & -7 & 6 \\ 8 & -6 & 5 \end{bmatrix}$

(2) Dengan menggunakan Minor-Kofaktor

Menentukan invers matriks dengan Minor-kofaktor ini, dilakukan dengan menggunakan konsep determinan (dilambangkan dengan *det*) dan konsep adjoint (dilambangkan dengan *adj*).

Misalkan A = $\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$ maka langkah-langkah menentukan invers matriks dengan metoda ini adalah sebagai berikut :

1. Menentukan minor matriks A untuk baris p dan kolom q (M_{pq})

$M_{11} = \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} = b_2c_3 - c_2b_3$ $\qquad M_{12} = \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} = a_2c_3 - c_2a_3$

$M_{13} = \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} = a_2b_3 - b_2a_3$ $\qquad M_{21} = \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} = b_1c_3 - c_1b_3$

$M_{22} = \begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} = a_1c_3 - c_1a_3$ $\qquad M_{23} = \begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix} = a_1b_3 - b_1a_3$

$M_{31} = \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} = b_1c_2 - c_1b_2$ $\qquad M_{32} = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} = a_1c_2 - c_1a_2$

$M_{33} = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - b_1a_2$

2. Menentukan kofaktor matriks A

   Kofaktor matriks A baris ke-p kolam ke-q dilambangkan C_{pq} ditentukan dengan rumus :

$$C_{pq} = (-1)^{p+q}\ M_{pq}$$

   Sehingga diperoleh matriks kofaktor C sebagai berikut :

$$C = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{22} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

3. Menentukan determinan matriks A

Determinan matriks A ditulis det(A) atau $|A|$ ditentukan dengan rumus:

$$\det(A) = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1.b_2.c_3 + b_1.c_2.a_3 + c_1.a_2.b_3 - c_1.b_2.a_3 - a_1.c_2.b_3 - b_1.a_2.c_3$$

atau dengan menggunakan kofaktor C_{pq} dengan rumus :

$\det(A) = a_1 C_{11} - b_1 C_{12} + c_1 C_{13}$

$\det(A) = a_2 C_{21} - b_2 C_{22} + c_2 C_{23}$

$\det(A) = a_3 C_{31} - b_3 C_{32} + c_3 C_{33}$

4. Menentukan matriks adjoint A, yakni transpose dari kofaktor matriks A, atau dirumuskan :

$$\text{Adj } A = C^t$$

5. Menentukan invers matriks A dengan rumus :

$$A^{-1} = \frac{1}{\det(A)} \text{ adj } A$$

Untuk lebih jelasnya ikutilah contoh soal berikut ini :

03. Tentukanlah Determinan matriks $A = \begin{bmatrix} 2 & 1 & -2 \\ -1 & 2 & 3 \\ 0 & 3 & 1 \end{bmatrix}$

Jawab

$$\left|\begin{array}{ccc|cc} 2 & 1 & -2 & 2 & 1 \\ -1 & 2 & 3 & -1 & 2 \\ 0 & 3 & 1 & 0 & 3 \end{array}\right|$$

det = (2)(2)(1) + (1)(3)(0) + (–2)(–1)(3) – (–2)(2)(0) – (2)(3)(3) – (1)(–1)(1)

det = 4 + 0 + 6 – 0 – 18 + 1

det = –7

04. Dengan menggunakan kofaktor, tentukanlah invers matriks $A = \begin{bmatrix} 3 & -5 & 0 \\ 2 & -3 & 1 \\ -1 & 2 & 2 \end{bmatrix}$

Jawab

Langkah 1 (*menentukan minor matriks*)

$M_{11} = \begin{vmatrix} -3 & 1 \\ 2 & 2 \end{vmatrix}$ = (–3)(2) – (1)(2) = –6 – 2 = –8

$M_{12} = \begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix}$ = (2)(2) – (1)(–1) = 4 + 1 = 5

$M_{13} = \begin{vmatrix} 2 & -3 \\ -1 & 2 \end{vmatrix}$ = (2)(2) – (–3)(–1) = 4 – 3 = 1

$M_{21} = \begin{vmatrix} -5 & 0 \\ 2 & 2 \end{vmatrix}$ = (–5)(2) – (0)(2) = –10 – 0 = –10

$M_{22} = \begin{vmatrix} 3 & 0 \\ -1 & 2 \end{vmatrix}$ = (3)(2) – (0)(–1) = 6 – 0 = 6

$M_{23} = \begin{vmatrix} 3 & -5 \\ -1 & 2 \end{vmatrix}$ = (3)(2) – (–5)(–1) = 6 – 5 = 1

$M_{31} = \begin{vmatrix} -5 & 0 \\ -3 & 1 \end{vmatrix}$ = (–5)(1) – (0)(–3) = –5 – 0 = –5

$M_{32} = \begin{vmatrix} 3 & 0 \\ 2 & 1 \end{vmatrix}$ = (3)(1) – (0)(2) = 3 – 0 = 3

$M_{33} = \begin{vmatrix} 3 & -5 \\ 2 & -3 \end{vmatrix}$ = (3)(–3) – (–5)(2) = –9 + 10 = 1

Langkah 2 (*menentukan kofaktor matriks*)

$C_{11} = (-1)^{1+1}\ M_{11}$ = (1)(–8) = –8

$C_{12} = (-1)^{1+2}\ M_{12}$ = (–1)(5) = –5

$C_{13} = (-1)^{1+3}\ M_{13}$ = (1)(1) = 1

$C_{21} = (-1)^{2+1}\ M_{21}$ = (–1)(–10) = 10

$C_{22} = (-1)^{2+2}\ M_{22}$ = (1)(6) = 6

$C_{23} = (-1)^{2+3}\ M_{23}$ = (–1)(1) = –1

$C_{31} = (-1)^{3+1}\ M_{31}$ = (1)(1) = 1

$C_{32} = (-1)^{3+2}\ M_{32}$ = (–1)(3) = –3

$C_{33} = (-1)^{3+3}\ M_{33}$ = (1)(1) = 1

Matriks kofaktornya : $C = \begin{bmatrix} -8 & -5 & 1 \\ 10 & 6 & -1 \\ 1 & -3 & 1 \end{bmatrix}$

Langkah 3 (*menentukan Determinan matriks*)

Menggunakan ekspansi baris pertama

det(A) = $a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$ = (3)(–8) + (–5)(–5) + (0)(1) = 1

Langkah 4 (*menentukan Adjoint matriks*)

Matiks kofaktor $C = \begin{bmatrix} -8 & -5 & 1 \\ 10 & 6 & -1 \\ 1 & -3 & 1 \end{bmatrix}$ adjoin nya adj(A) = $\begin{bmatrix} -8 & 10 & 1 \\ -5 & 6 & -3 \\ 1 & -1 & 1 \end{bmatrix}$

Langkah 4 (*menentukan Invers matriks*)

$A^{-1} = \frac{1}{\det(A)}$ adj(A)

$$A^{-1} = \frac{1}{1}\begin{bmatrix} -8 & 10 & 1 \\ -5 & 6 & -3 \\ 1 & -1 & 1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} -8 & 10 & 1 \\ -5 & 6 & -3 \\ 1 & -1 & 1 \end{bmatrix}$$

05. Dengan menggunakan kofaktor, tentukanlah invers matriks A = $\begin{bmatrix} 3 & -5 & 0 \\ 2 & -3 & 1 \\ -1 & 2 & 2 \end{bmatrix}$

Jawab

Langkah 1 (*menentukan minor matriks*)

$M_{11} = \begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix}$ = (2)(1) – (1)(3) = 2 – 3 = –1

$M_{12} = \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}$ = (1)(1) – (1)(1) = 1 – 1 = 0

$M_{13} = \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix}$ = (1)(3) – (2)(1) = 3 – 2 = 1

$M_{21} = \begin{vmatrix} 3 & 3 \\ 3 & 1 \end{vmatrix}$ = (3)(1) – (3)(3) = 3 – 9 = –6

$M_{22} = \begin{vmatrix} 1 & 3 \\ 1 & 1 \end{vmatrix}$ = (1)(1) – (3)(1) = 1 – 3 = –2

$M_{23} = \begin{vmatrix} 1 & 3 \\ 1 & 3 \end{vmatrix}$ = (1)(3) – (1)(3) = 3 – 3 = 0

$M_{31} = \begin{vmatrix} 3 & 3 \\ 2 & 1 \end{vmatrix}$ = (3)(1) – (3)(2) = 3 – 6 = –3

$M_{32} = \begin{vmatrix} 1 & 3 \\ 1 & 1 \end{vmatrix}$ = (1)(1) – (3)(1) = 1 – 3 = –2

$M_{33} = \begin{vmatrix} 1 & 3 \\ 1 & 2 \end{vmatrix}$ = (1)(2) – (3)(1) = 2 – 3 = –1

Langkah 2 (*menentukan kofaktor matriks*)

$C_{11} = (-1)^{1+1}\ M_{11}$ = (1)(–1) = –1

$C_{12} = (-1)^{1+2}\ M_{12}$ = (–1)(0) = 0

$C_{13} = (-1)^{1+3}\ M_{13}$ = (1)(1) = 1

$C_{21} = (-1)^{2+1}\ M_{21}$ = (–1)(–6) = 6

$C_{22} = (-1)^{2+2}\ M_{22}$ = (1)(2) = 2

$C_{23} = (-1)^{2+3}\ M_{23}$ = (–1)(0) = 0

$C_{31} = (-1)^{3+1}\ M_{31}$ = (1)(–3) = –3

$C_{32} = (-1)^{3+2}\ M_{32}$ = (–1)(–2) = 2

$C_{33} = (-1)^{3+3}\ M_{33}$ = (1)(–1) = –1

Matriks kofaktornya : C = $\begin{bmatrix} -1 & 0 & 1 \\ 6 & 2 & 0 \\ -3 & 2 & -1 \end{bmatrix}$

Langkah 3 (*menentukan Determinan matriks*)

Menggunakan ekspansi baris pertama

det(A) = $a_{11}C_{11}$ + $a_{12}C_{12}$ + $a_{13}C_{13}$ = (1)(–1) + (3)(0) + (3)(1) = –1 + 0 + 3 = 2

Langkah 4 (*menentukan Adjoint matriks*)

Matiks kofaktor C = $\begin{bmatrix} -1 & 0 & 1 \\ 6 & 2 & 0 \\ -3 & 2 & -1 \end{bmatrix}$ adjoin nya adj(A) = $\begin{bmatrix} -1 & 6 & -3 \\ 0 & 2 & 2 \\ 1 & 0 & -1 \end{bmatrix}$

Langkah 4 (*menentukan Invers matriks*)

$A^{-1} = \dfrac{1}{\det(A)}$ adj(A)

$$A^{-1} = \frac{1}{2}\begin{bmatrix} -1 & 6 & -3 \\ 0 & 2 & 2 \\ 1 & 0 & -1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} -1/2 & 3 & -3/2 \\ 0 & 1 & 1 \\ 1/2 & 0 & -1/2 \end{bmatrix}$$